Higher Order Shear Deformation Plate Theory

by

Gjermund Mæsel Kolvik

THESIS

for the degree of

MASTER OF SCIENCE

(Master i Anvendt matematikk og mekanikk)

Faculty of Mathematics and Natural Sciences
University of Oslo

May 2012

Det matematisk- naturvitenskapelige fakultet
Universitetet i Oslo

Preface

This thesis has been written to fulfill the degree of Master of Science at the University of Oslo, Department of Mathematics, Mechanics Division.

Professors Noël Challamel and Jostein Hellesland at the University of Oslo have been my supervisors during this project. Professor Challamel's expertise on higher order theories has been of immense help. I would like to thank him for his motivating enthusiasm and welcoming presence. I feel fortunate to have made his acquaintance during his year at the University of Oslo.

I would like to express my gratitude to professor Jostein Hellesland for his excellent guidance and advice throughout the project. His vast knowledge in solid mechanics and encouraging support is highly appreciated.

Abstract

Several plate theories have been developed to describe the static and dynamic behaviour of plates. This thesis is predominantly a study of plate theories including shear effects, with emphasis on higher order shear deformation theories. The plate theories of Reddy and Shi are specifically analysed. An effort towards the development of a unified higher order shear deformation plate theory is presented in this thesis.

The buckling behaviour of some generic higher order shear plate models is investigated in a unified framework. The governing equations of the buckling problem are obtained from a variational approach, leading to generic partial differential equations and associated boundary conditions. Buckling problems are analytically solved using the Navier method on isotropic simply supported plates under uniform in-plane loads. Buckling load relationships between classical plate theory and the plate theories including shear effects are also investigated. The accuracy of the unified shear deformation theory is demonstrated through these buckling results.

The numerical results of the buckling problems, indicate that the theories of Reddy and Shi yield exactly the same buckling loads for the problems in question, whereas the buckling loads estimated from some other higher order theories vary slightly. Due to the simple nature of the solved buckling problems, in terms of geometrical and material properties, all the higher order theories yield almost the same buckling loads as the first order shear deformation theory. This coincides with the fact that higher order plate theories have their advantages when being used for laminated composite plates.

It is the author's belief that the unified higher order shear deformation plate theory presented in this thesis, can contribute to gathering many of the higher order theories presented in the literature in a common framework.

Contents

Chapter 1

Introduction

1.1 Background and motivation

Plate structures are major load carrying elements in structural mechanics, both in aeronautics, on land and in naval engineering. Such plates are often subjected to significant in plane compression forces and/or shear loading. Various plate theories are available to describe the static and dynamic behaviour of such plates. Depending on the plate geometry and material properties, it is of interest to utilize one plate theory over another. Understanding the differences between the theories, and the application of them, is of interest both to engineers working in the field of plate structures, as well as researchers working with the development of new knowledge on plates.

Since the middle of the 19th century there has been ongoing research and development of plate theories. This research has resulted in three main categories in the field of plate theories:

- Approximately 1850: Kirchhoff plate theory, classical plate theory (CPT). Suitable for thin plates with thickness to width ratio less than 1/10. Neglects shear effects.
- Approximately 1950: Mindlin plate theory, first order shear deformation plate theory (FSDT). Suitable for thick plates with thickness to width ratio more than 1/10. Includes shear effects.
- Approximately 1980: Higher order shear deformation plate theories (HSDT). Can represent the kinematics better than FSDT and are especially suitable for composite plates. Includes shear effects.

A few years prior to Mindlin, Reissner developed a plate theory including shear effects and therefor suitable for thick plates. The theories of Mindlin and Reissner are similar, and they are often referred to as one Reissner-Mindlin plate theory. According to Wang et al. [12] this is a misleading description as the two theories are based on different assumptions. In this thesis we do not treat the theory of Reissner, and when reference is made to FSDT we refer to Mindlin plate theory only.

Many higher order shear deformation theories have been developed in the last 30 years. Without further comment we mention the theories of Touratier (transverse strain distribution as a sine function), Soldatos (hyperbolic shear deformation theory), Mechab (hyperbolic shear deformation theory), Karama et al. (exponential variation for the transverse strain) in addition to Reddy, and Shi. The latter two proposed a parabolic variation of the transverse shear strain [2].

Among the higher order plate theories, the one of J. N. Reddy is considered to be the most popular theory used for analysis of laminated composite plates [10]. In 2007 Guangyu Shi presented a new shear deformation theory of plates, similar to Reddy's in the sense that they both proposed a parabolic variation of the transverse shear strain. Both Reddy's and Shi's theories are third order plate theories, meaning that the displacement field is assumed to be described by a function of third order.

The motivation behind this thesis is to describe several of the above mentioned higher order theories in a unified framework, emphasizing the theories of Reddy and Shi.

In 2011 Challamel [4] presented an article, in which he analytically studied and treated the buckling problem of a third order shear beam-column in the framework of gradient elasticity Timoshenko beam theory. This thesis is a similar study in the field of plate theories.

1.2 Objectives and Scope

The main objective of the thesis is to investigate the buckling behaviour of some generic higher order shear plate models in a unified framework. In order to achieve that, efforts will be made to show that most higher order shear plate models developed in the literature, whatever shear strain distribution assumptions over the cross section, can be classified in a common gradient elasticity first order shear plate theory.

A short review of CPT and FSDT will be given, but the thesis will primarily focus on the two plate theories of Reddy and Shi. Based on the displacement field that the two theories propose, a unified system of partial differential equations that describe both theories will be developed. Variational consistent boundary conditions will also be presented as they appear when using the principle of minimum potential energy, but they will not be thoroughly investigated. The differential equations will be used to solve the buckling problem for isotropic simply supported plates under uniform in-plane compressive load on all edges.

Even though this study primarily considers the two theories of Reddy and Shi, and only solves fairly basic buckling problems (simply supported, isotropic plates of homogenous material), it is the author's belief that the equations developed in this thesis can be transferred to solve more complex buckling problems (orthotropic, laminated composite plates), and describe several higher order shear plate theories. In that respect, this study represents an effort towards the development of a unified higher order shear deformation plate theory.

1.3 Outline of the report

Chapter 2 gives a background frame. A review of different plate theories is presented in Section 2.1, and central methods for deriving the differential equations and boundary conditions, as well as a description of the Navier method for solving buckling problems are presented in Section 2.2.

Chapter 3 deals with FSDT. The methodology presented in this chapter will be used also when dealing with HSDT, and the equations derived for FSDT give a basis for comparison with HSDT.

Chapters 4 and 5 contain the theory and relationships linked with HSDT. Chapter 4 presents HSDT in the light of gradient elasticity, whereas the differential equations for HSDT are derived and used to solve the buckling problem in Chapter 5.

In Chapter 6 the buckling results are presented for CPT, FSDT and HSDT in tables.

Chapter 7 contains conclusion and suggestions for further work.

Chapter 2

Development of plate theories

2.1 Short background

Several plate theories have been developed to describe the static and dynamic behaviour of plates, and many of them are based on displacement approximations. According to Altenbach [1] engineers prefer theories which are based on hypotheses. The first theory of plates based on displacement assumptions, was presented by Kirchhoff in 1850. Kirchhoff's assumptions read as follows [13]:

- Straight lines perpendicular to the mid-surface (i.e., transverse normals) before deformation remain straight after deformation.
- The transverse normals do not experience elongation (i.e., they are inextensible).
- The transverse normals rotate such that they remain perpendicular to the mid-surface after deformation.

The consequence of the Kirchhoff hypothesis is that the transverse shear strains are zero, and consequently, the transverse stresses do not enter the theory. The theory is known as the Kirchhoff plate theory and it is an extension of Euler-Bernoulli beam theory. Often Kirchhoff plate theory is referred to as the classical plate theory. It does not include shear effects and is therefore applicable to thin plates only. The classical plate theory will give erroneous results when being used for thick plates, especially plates made of advanced composites [6].

To account for the transverse shear strains, shear deformation plate theories have been developed. Mindlin proposed his theory in 1951, which is an extension of Kirchhoff plate theory. In Mindlin plate theory the basic equations are derived by assumption that the in-plane displacements are linearly distributed across the plate thickness. This leads to the transverse shear stresses being constant across the plate thickness, so the zero shear stress condition on the plate face is not satisfied. This forces the use of shear correction factors, comparable to the need for shear correction factors in the Timoshenko beam theory. Mindlin plate theory is often referred to as first order shear deformation plate theory, and it has been extensively used in the analysis of shear flexible plates and shells. But when Mindlin plate theory is applied to composite plates, the difficulty in accurately evaluating the shear correction factors presents the shortcommings of FSDT [10].

To properly approximate the nonlinear distribution of transverse shear strains along the plate thickness, quite a number of higher order shear deformation plate theories were developed. Such HSDTs have proven to be highly applicable to laminated composite plates. Levinson and Murthy developed plate theories that employ three-order polynomials to expand the in plane displacement across the plate thickness, which in turn excludes the need for shear correction factors. However, in Levinson's and Murthy's plate theories they used the equilibrium equations of the classical plate theory, which is variationally inconsistent with the kinematics of displacements. In order to rectify this defect, Reddy presented his plate theory in 1984, which developes variational consistent equilibrium equations for plates [6].

The two-dimensional plate theories with higher order in plane displacements but a constant deflection through the plate thickness are the so called simple higher order shear deformation theories [10]. A two dimensional structure is here deifined as [1]:

A two dimensional load bearing structural element is a model for analysis in Engineering/Structural Mechanics, having two geometrical dimensions which are of the same order and which are significantly larger in comparison with the third (thickness) direction.

Reddy's plate theory is known for being the most popular simple HSDT used for composite plate analysis [10]. According to Liu [6], another HSDT was developed by Ambartsumian in which he proposed another transverse shear stress function in order to explain deformation of layered anisotropic plates. Reddy's and Ambartsumian's theories formed a solid benchmark for the development of a new simple HSDT. In 2007 Shi presented a new HSDT which is developed on the basis of Murthy's and Reddy's theories. Shi derived a new set of variational consistent governing equations and associated proper boundary conditions. Both Reddy and Shi employed third order polynomials in the expansion of the displacement components through the thickness of the plate, leading to a parabolic variation of the transverse shear stresses. They are therefore sometimes referred to as *Third order shear deformation theories* or *Parabolic shear deformation theories*.

It should be remarked that it is not preferable to always use HSDT in order to get as accurate numerical results as possible. On the contrary, Reddy [9] expresses that: *Higher-order theories can represent the kinematics better, may not require shear correction factors, and can yield more accurate interlaminar stress distributions. However, they involve higher-order stress resultants that are difficult to interpret physically and require considerably more computational effort. Therefore, such theories should be used only when necessary.*

Figure 2.1 shows an illustration of how CPT, FSDT and HSDT differ from eachother in terms of in-plane displacements. Wang et al. presented this figure in [11].

Aydogdu [2] presented in 2006 a study in which he compared various HSDTs with available three-dimensional analysis. He showed that while the transverse displacement and the stresses are best predicted by the exponential shear deformation theory (Karama et al.), the parabolic shear deformation (Reddy) and the hyperbolic shear deformation (Soldatos) theories yield more accurate predictions for the natural frequencies and the buckling loads.

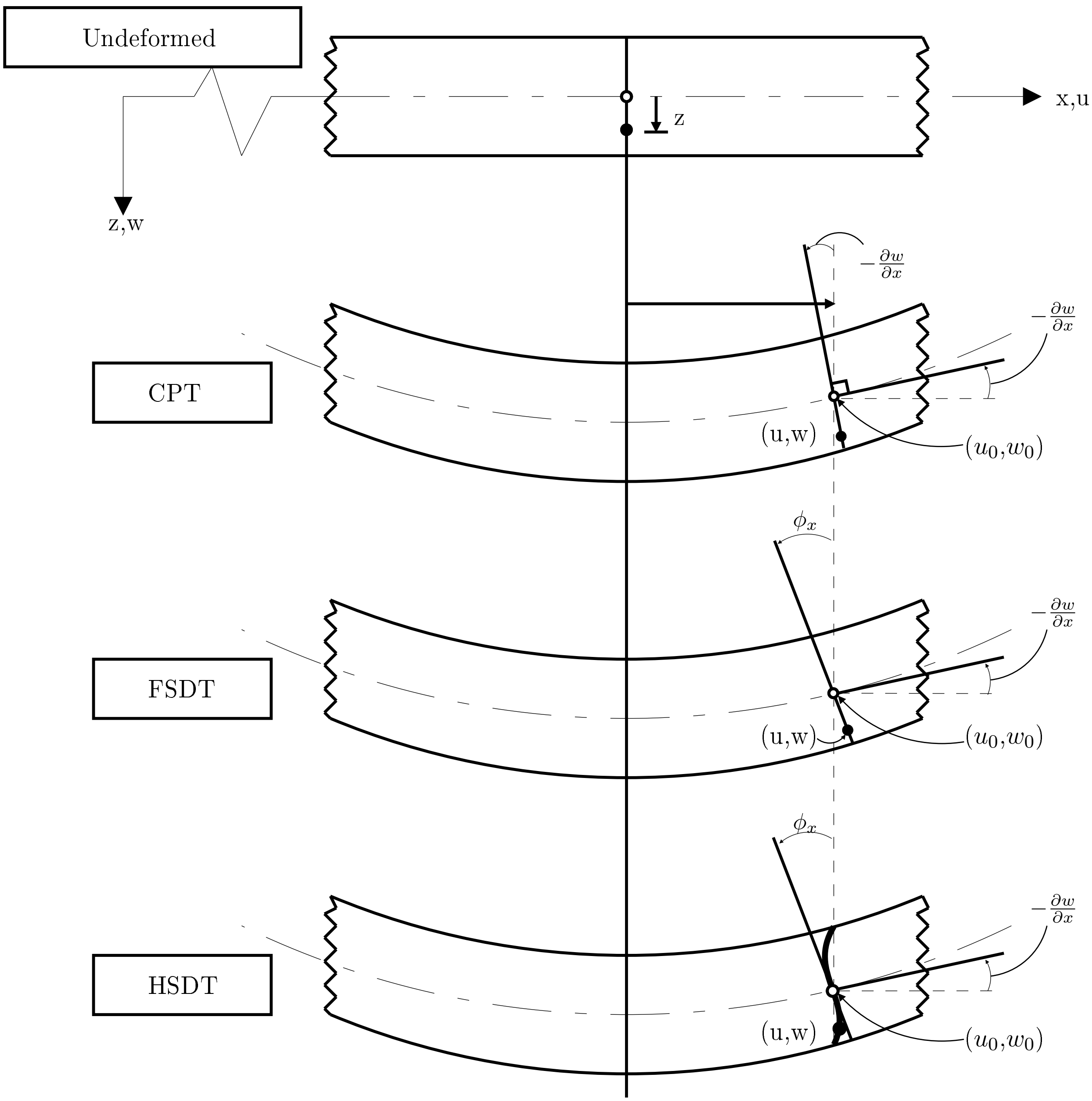

Figure 2.1: Undeformed and deformed geometries of an edge of a plate in various plate theories. u_0 denotes the in-plane displacement.

2.2 Theoretical preliminaries

2.2.1 The principle of minimum potential energy (PMPE)

Variational principles can be used to obtain governing differential equations and associated boundary conditions. PMPE is a special case of the principle of virtual displacements that deals with linear as well as nonlinear elastic bodies [9]. For elastic bodies (in the absence of temperature variations) there exists a *strain energy* U and a *potential of external forces* V.

The sum $U + V = \Pi$ is called the total potential energy of the elastic body.

In general the potential energy expresses the potential or the ability of a body to perform work [3]. Let us consider an arbitrary body under the impact of traction and volume forces, with given boundary conditions, see Figure 2.2.

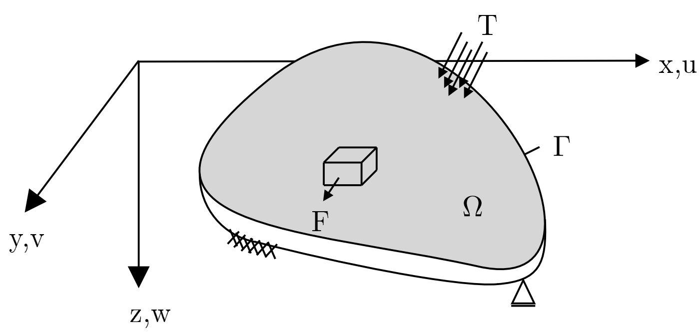

Figure 2.2: Arbitrary body in equilibrium.

In the cartesian coordinate system the volume force at a point is given by the components

$$\mathbf{F} = \{F_x \quad F_y \quad F_z\}^T \qquad \text{(Force per unit volume)} \tag{2.1}$$

The traction force at a surface point is given by the componets

$$\mathbf{T} = \{T_x \quad T_y \quad T_z\}^T \qquad \text{(Force per unit area)} \tag{2.2}$$

The displacement of an arbitrary point is a vector that can be decomposed along the three coordinate axes

$$\mathbf{u} = \{u \quad v \quad w\}^T \tag{2.3}$$

The potential of external forces acting on the body is given by

$$\begin{aligned} V &= -\int_{\Omega} (F_x u + F_y v + F_z w)\, dv \\ &\quad - \int_{\Gamma_\sigma} (T_x u + T_y v + T_z w)\, ds \\ &= -\int_{\Omega} \mathbf{u}^T \mathbf{F} dv - \int_{\Gamma_\sigma} \mathbf{u}^T \mathbf{T} ds \end{aligned} \tag{2.4}$$

where Ω is the volume of the body and Γ_σ is the part of the surface where the traction force is located. dv and ds denote the volume and surface elements of Ω.

The energy that is stored in the body due to deformation is called the strain energy. To find a proper expression for the strain energy it is necessary to know the material's stress-strain relationship. For a linear elastic material the strain energy is simply found from the stress strain diagram, see Figure 2.3.

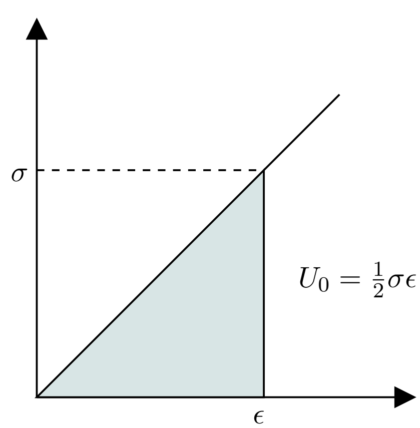

Figure 2.3: Strain energy density in the linear elastic case.

The strain energy stored in a volume unit dv, is called *the strain energy density* and it is given by

$$U_0 = \int_0^{\epsilon} \sigma^T d\epsilon \tag{2.5}$$

For a linear elastic material the strain energy density is

$$
\begin{aligned}
U_0 =& \frac{1}{2}\left(\sigma_x\epsilon_x + \sigma_y\epsilon_y + \sigma_z\epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz} +\right) \\
=& \frac{1}{2}\epsilon^T\sigma = \frac{1}{2}\epsilon^T\mathbf{C}\epsilon
\end{aligned}
\tag{2.6}
$$

where $\mathbf{C}$ expresses the generalized Hooke's law:

$$
\sigma = \mathbf{C}\epsilon \tag{2.7}
$$

The strain energy is now found by integrating the strain energy density over the volume

$$
U = \int_\Omega U_0 dv \tag{2.8}
$$

PMPE states that the deflections in a body in static equilibrium, will settle in such a way that the total potential energy is a minimum. This results in the derivative of Π with respect to the displacements, must equal zero.

$$
\partial U + \partial V = \partial(U + V) \equiv \partial\Pi = 0 \tag{2.9}
$$

In other words, it means that of all admisssible displacements, those which satisfy static equlibrium make the total potential energy a minimum:

$$
\Pi(u) \leq \Pi(\bar{u}) \tag{2.10}
$$

where u is the true solution and $\bar{u}$ is any admissible displacement field. The equality holds only if $u = \bar{u}$.

2.2.2 The divergence theorem

When finding the total potential energy of a body we need to integrate the expressions for the strain energy and the potential of external forces. In that operation we use the divergence theorem to obtain the differential equations and associated boundary conditions. The divergence theorem is explained as follows [9].

Let Ω denote a region in space sorrounded by the surface Γ, and let ds be a differential element of the surface whose unit outward normal is denoted by $\bar{n}$.

Let dv be a differential volume element and A a vector function defined over the region Γ. Then the following integral identity holds:

$$\int_{\Omega} \frac{\partial A_i}{\partial x_i} dv = \oint_{\Gamma} n_i A_i ds \tag{2.11}$$

where a circle on the integral sign signifies integration over the total boundary.

2.2.3 Navier method

There are several approaches to solving a buckling problem. In this study we use the Navier method. According to Reddy [7] the Navier solutions can be developed for a rectangular plate (or laminate) when all four edges are simply supported. Other methods frequently used for solving plate buckling problems are the ones of Levy and Rayleigh-Ritz. Levy's solutions can be developed for plates with two opposite edges simply supported and the remaining two edges having any possible combination of boundary conditions: free, simple support, or fixed support. The Rayleigh-Ritz method can be used to determine approximate solutions for more general bounday conditions.

In Navier's method the generalized diplacements are expanded in a double trigonometric series in terms of unknown parameters. The choice of the functions in the series is restricted to those which satisfy the boundary conditions of the problem. Substitution of the displacement expansions into the governing equations should result in a unique, invertible, set of algebraic equations among the parameters of the expansion. Otherwise, the Navier solution can not be developed for the problem [7].

Chapter 3

Mindlin plate theory

Before treating HSDT we show in this chapter how to obtain the differential equations and boundary conditions in FSDT. The methodology will be the same in both cases and can be summarized as:

- Start from the displacement field assumptions.
- Consider the stress strain relationship for a linear elastic material.
- Find the expressions for U and V.
- Use the principle of minimum potential energy to obtain the governing equations and associated boundary conditions.
- Solve the buckling problem by using the Navier method.

The objective is to obtain a system of three partial differential equations by investigating the minimum potential energy of the plate subjected to uniform in-plane loads. We want to find the first variation of the strain energy and the potential energy of external forces, and use the principle of minimum potential energy to obtain the differential equations and associated boundary conditions. That is to say we need to solve

$$\delta\Pi = \delta U + \delta V = 0 \tag{3.1}$$

3.1 Strain energy

The Mindlin plate theory is based on the displacement field

$$\left.\begin{aligned} u &= z\phi_x \\ v &= z\phi_y \\ w &= w \end{aligned}\right\} \tag{3.2}$$

where (u,v,w) are the displacement components along the (x,y,z) coordinate directions, respectivly. ϕ_x and ϕ_y denote rotations about the y and x axes, respectivly.

In view of that displacement field the strains are given by

$$\left.\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} = z\frac{\partial \phi_x}{\partial x} \\ \epsilon_y &= \frac{\partial v}{\partial y} = z\frac{\partial \phi_y}{\partial y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) \\ \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \phi_x + \frac{\partial w}{\partial x} \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \phi_y + \frac{\partial w}{\partial y} \end{aligned}\right\} \tag{3.3}$$

Assuming the plate material is isotropic and obeys Hooke's law the stress strain relationship is given by

$$\left.\begin{aligned} \sigma_x &= \frac{E}{1-\nu^2}\left(\epsilon_x + \nu\epsilon_y\right) \\ \sigma_y &= \frac{E}{1-\nu^2}\left(\epsilon_y + \nu\epsilon_x\right) \\ \sigma_{xy} &= G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} \\ \sigma_{xz} &= G\gamma_{xz} = \frac{E}{2(1+\nu)}\gamma_{xz} \\ \sigma_{yz} &= G\gamma_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz} \end{aligned}\right\} \tag{3.4}$$

This can be written in matrix format as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 & 0 & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 & 0 & 0 \\ 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \tag{3.5}$$

where E is the Young's modulus, ν is Poisson's ratio and the shear modulus G is related to E and ν by $G = \frac{E}{2(1+\nu)}$.

In general the strain energy for a linear elastic material can be written, see Eqs. (2.6) and (2.8)

$$U = \frac{1}{2} \int\int\int \left(\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right) dxdydz \tag{3.6}$$

Since we have plane stress $\sigma_z \epsilon_z$ is excluded.

Introducing the stress strain relationship into Eq. (3.6) gives

$$U = \frac{E}{2\left(1-\nu^2\right)} \int\int\int \left(\epsilon_x^2 + \epsilon_y^2 + 2\nu\epsilon_x\epsilon_y + \frac{(1-\nu)}{2}\left(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2\right)\right) dxdydz \tag{3.7}$$

Introducing the strain expressions from Eq. (3.3) into Eq. (3.7) gives

$$\begin{aligned} U =& \frac{E}{2\left(1-\nu^2\right)} \int\int\int z^2 \left(\frac{\partial \phi_x}{\partial x}\right)^2 + z^2 \left(\frac{\partial \phi_y}{\partial y}\right)^2 + 2\nu z^2 \left(\frac{\partial \phi_x}{\partial x}\right)\left(\frac{\partial \phi_y}{\partial y}\right) \\ &+ \frac{(1-\nu)}{2}\left(z^2\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)^2 + \left(\phi_x + \frac{\partial w}{\partial x}\right)^2 + \left(\phi_y + \frac{\partial w}{\partial y}\right)^2\right) dxdydz \end{aligned} \tag{3.8}$$

Now integrating Eq. (3.8) with respect to z gives

$$\begin{aligned} U =& \frac{D}{2} \int\int \left[\left(\frac{\partial \phi_x}{\partial x}\right)^2 + \left(\frac{\partial \phi_y}{\partial y}\right)^2 + 2\nu\left(\frac{\partial \phi_x}{\partial x}\frac{\partial \phi_y}{\partial y}\right) + \frac{(1-\nu)}{2}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)^2\right] dxdy \\ &+ \frac{Eh}{4\left(1+\nu\right)} \int\int \left[\left(\phi_x + \frac{\partial w}{\partial x}\right)^2 + \left(\phi_y + \frac{\partial w}{\partial y}\right)^2\right] dxdy \end{aligned} \tag{3.9}$$

where the plate's bending stiffness

$$D = \frac{E}{1-\nu^2} \int_{-h/2}^{h/2} z^2 dz = \frac{Eh^3}{12\,(1-\nu^2)}$$

and h is the plate thickness.

Eq. (3.9) can be written in matrix format as

$$U = \frac{1}{2} \int \int \Big\{ \phi_{x,x} \quad \phi_{y,y} \quad (\phi_{x,y} + \phi_{y,x}) \quad (w_{,x} + \phi_x) \quad (w_{,y} + \phi_y) \Big\} \cdot$$

$$\begin{bmatrix} D_{11} & D_{12} & 0 & 0 & 0 \\ D_{12} & D_{22} & 0 & 0 & 0 \\ 0 & 0 & D_{66} & 0 & 0 \\ 0 & 0 & 0 & A_{44} & 0 \\ 0 & 0 & 0 & 0 & A_{55} \end{bmatrix} \cdot \begin{Bmatrix} \phi_{x,x} \\ \phi_{y,y} \\ (\phi_{x,y} + \phi_{y,x}) \\ (w_{,x} + \phi_x) \\ (w_{,y} + \phi_y) \end{Bmatrix} dxdy \tag{3.10}$$

where in the isotropic case

$$\left.\begin{aligned} D_{11} &= \frac{Eh^3}{12(1-\nu^2)} = D \\ D_{12} &= \nu D \\ D_{22} &= D \\ D_{66} &= \frac{(1-\nu)}{2} D \\ A_{44} &= A_{55} = \frac{Eh}{2\,(1+\nu)} \end{aligned}\right\} \tag{3.11}$$

Using variational calculus gives the first variation of the strain energy

$$
\begin{aligned}
\delta U = D \int\int \Bigg[& \left(\frac{\partial \phi_x}{\partial x} + \nu \frac{\partial \phi_y}{\partial y} \right) \delta \left(\frac{\partial \phi_x}{\partial x} \right) \\
& + \left(\frac{\partial \phi_y}{\partial y} + \nu \frac{\partial \phi_x}{\partial x} \right) \delta \left(\frac{\partial \phi_y}{\partial y} \right) \\
& + \left(\frac{(1-\nu)}{2} \frac{\partial \phi_x}{\partial y} + \frac{(1-\nu)}{2} \frac{\partial \phi_y}{\partial x} \right) \delta \left(\frac{\partial \phi_x}{\partial y} \right) \\
& + \left(\frac{(1-\nu)}{2} \frac{\partial \phi_x}{\partial y} + \frac{(1-\nu)}{2} \frac{\partial \phi_y}{\partial x} \right) \delta \left(\frac{\partial \phi_y}{\partial x} \right) \Bigg] dxdy \\
+ \frac{Eh}{2(1+\nu)} \int\int \Bigg[& \left(\phi_x + \frac{\partial w}{\partial x} \right) \delta \phi_x + \left(\phi_x + \frac{\partial w}{\partial x} \right) \delta \left(\frac{\partial w}{\partial x} \right) \\
& + \left(\phi_y + \frac{\partial w}{\partial y} \right) \delta \phi_y + \left(\phi_y + \frac{\partial w}{\partial y} \right) \delta \left(\frac{\partial w}{\partial y} \right) \Bigg] dxdy \qquad (3.12)
\end{aligned}
$$

The plate constitutive equations are given by

$$M_{xx} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x z dz = D \left(\frac{\partial \phi_x}{\partial x} + \nu \frac{\partial \phi_y}{\partial y} \right) \qquad (3.13a)$$

$$M_{yy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y z dz = D \left(\nu \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} \right) \qquad (3.13b)$$

$$M_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xy} z dz = \frac{D(1-\nu)}{2} \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right) \qquad (3.13c)$$

$$Q_x = \kappa \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xz} dz = \frac{\kappa E h}{2(1+\nu)} \left(\phi_x + \frac{\partial w}{\partial x} \right) \qquad (3.13d)$$

$$Q_y = \kappa \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yz} dz = \frac{\kappa E h}{2(1+\nu)} \left(\phi_y + \frac{\partial w}{\partial y} \right) \qquad (3.13e)$$

where κ is the shear correction factor.

Since the transverse shear stresses are represented as constant through the plate thickness in FSDT, we introduce κ to modify the transverse shear stresses. It is a well known fact that the transverse shear stresses are parabolic through the plate thickness [11]. It is normal to introduce $\kappa = \frac{5}{6}$.

Introducing the plate constitutive equations into Eq. (3.12) gives

$$
\begin{aligned}
\delta U = \int\int \Bigg[& \left(M_{xx}\frac{\partial}{\partial x} + M_{xy}\frac{\partial}{\partial y} + Q_x \right) \delta\phi_x \\
& + \left(M_{yy}\frac{\partial}{\partial y} + M_{xy}\frac{\partial}{\partial x} + Q_y \right) \delta\phi_y \\
& + \left(Q_x\frac{\partial}{\partial x} + Q_y\frac{\partial}{\partial y} \right) \delta w \Bigg] dxdy
\end{aligned} \tag{3.14}
$$

Now integrating Eq. (3.14) by parts and using the divergence theorem we obtain

<u>From $\delta\phi_x$:</u>

$$
\begin{aligned}
& \int\int \left(M_{xx}\frac{\partial}{\partial x} + M_{xy}\frac{\partial}{\partial y} + Q_x \right) \delta\phi_x \; dxdy \\
= & \oint_\Gamma \left(M_{xx}n_x + M_{xy}n_y \right) \delta\phi_x \; ds \\
- & \int\int \left(M_{xx,x} + M_{xy,y} - Q_x \right) \delta\phi_x \; dxdy
\end{aligned} \tag{3.15}
$$

<u>From $\delta\phi_y$:</u>

$$
\begin{aligned}
& \int\int \left(M_{yy}\frac{\partial}{\partial y} + M_{xy}\frac{\partial}{\partial x} + Q_y \right) \delta\phi_y \; dxdy \\
= & \oint_\Gamma \left(M_{yy}n_y + M_{xy}n_x \right) \delta\phi_y \; ds \\
- & \int\int \left(M_{yy,y} + M_{xy,x} - Q_y \right) \delta\phi_y \; dxdy
\end{aligned} \tag{3.16}
$$

<u>From δw:</u>

$$
\begin{aligned}
& \int\int \left(Q_x\frac{\partial}{\partial x} + Q_y\frac{\partial}{\partial y} \right) \delta w \; dxdy \\
= & \oint_\Gamma \left(Q_x n_x + Q_y n_y \right) \delta w \; ds \\
- & \int\int \left(Q_{x,x} + Q_{y,y} \right) \delta w \; dxdy
\end{aligned} \tag{3.17}
$$

A comma followed by subscripts denotes differentiation with respect to the subscripts. For example $M_{xx,x} = M_{xx}\frac{\partial}{\partial x}$. n_x and n_y denote the direction cosines of the unit normal $\bar{n}$ on the boundary Γ. The same convention is used in [11]. That completes the first variation of the strain energy. We now turn to the potential energy of external forces and repeat the procedure.

3.2 Potential energy of external forces

The potential energy of external forces V is given by the forces acting on the boundary of the plate and the curvature, assuming small rotations. Note that V depends on the variable (δw) only

$$V = -\frac{1}{2}\int\int N_{xx}\left(\frac{\partial w}{\partial x}\right)^2 + N_{yy}\left(\frac{\partial w}{\partial y}\right)^2 + N_{xy}\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) dxdy \tag{3.18}$$

Variational calculus leads to

$$\delta V = -\int\int N_{xx}\frac{\partial w}{\partial x}\,\delta\frac{\partial w}{\partial x} + N_{yy}\frac{\partial w}{\partial y}\,\delta\frac{\partial w}{\partial y} \quad + N_{xy}\left(\frac{\partial w}{\partial y}\,\delta\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x}\,\delta\frac{\partial w}{\partial y}\right) dxdy \tag{3.19}$$

When integrating Eq. (3.19) by parts and using the divergence theorem we obtain

$$\begin{aligned} &-\int\int N_{xx}\frac{\partial w}{\partial x}\,\delta\frac{\partial w}{\partial x}dxdy \\ &= -\oint_\Gamma \left(N_{xx}\frac{\partial w}{\partial x}n_x\right)\delta w ds + \int\int N_{xx}\frac{\partial^2 w}{\partial x^2}\delta w dxdy \end{aligned} \tag{3.20}$$

$$\begin{aligned} &-\int\int N_{yy}\frac{\partial w}{\partial y}\,\delta\frac{\partial w}{\partial y}dxdy \\ &= -\oint_\Gamma \left(N_{yy}\frac{\partial w}{\partial y}n_y\right)\delta w ds + \int\int N_{yy}\frac{\partial^2 w}{\partial y^2}\delta w dxdy \end{aligned} \tag{3.21}$$

$$\begin{aligned} &-\int\int N_{xy}\left(\frac{\partial w}{\partial y}\,\delta\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x}\,\delta\frac{\partial w}{\partial y}\right) dxdy \\ &= -\oint_\Gamma \left[\left(N_{xy}\frac{\partial w}{\partial y}n_x\right) + \left(N_{xy}\frac{\partial w}{\partial x}n_y\right)\right]\delta w ds + \int\int 2N_{xy}\frac{\partial^2 w}{\partial x\partial y}\delta w dxdy \end{aligned} \tag{3.22}$$

Summarizing the results in Eqs. (3.20), (3.21) and (3.22) gives

$$\delta V = \int\int \left(N_{xx}\frac{\partial^2 w}{\partial x^2} + N_{yy}\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y}\right)\delta w dx dy$$
$$- \oint_\Gamma \left[\left(N_{xx}\frac{\partial w}{\partial x}n_x\right) + \left(N_{yy}\frac{\partial w}{\partial y}n_y\right) + \left(N_{xy}\frac{\partial w}{\partial y}n_x\right) + \left(N_{xy}\frac{\partial w}{\partial x}n_y\right)\right]\delta w ds \tag{3.23}$$

3.3 System of partial differential equations

We have now found the first variation of the strain energy and the potential energy of external forces. Inserting that into

$$\delta\Pi = \delta U + \delta V = 0 \tag{3.24}$$

gives

$$\delta\Pi = \oint_\Gamma (M_{xx}n_x + M_{xy}n_y)\,\delta\phi_x \, ds$$
$$- \int\int (M_{xx,x} + M_{xy,y} - Q_x)\,\delta\phi_x \, dxdy$$
$$+ \oint_\Gamma (M_{yy}n_y + M_{xy}n_x)\,\delta\phi_y \, ds$$
$$- \int\int (M_{yy,y} + M_{xy,x} - Q_y)\,\delta\phi_y \, dxdy$$
$$+ \oint_\Gamma (Q_x n_x + Q_y n_y) - \left[\left(N_{xx}\frac{\partial w}{\partial x}n_x\right) + \left(N_{yy}\frac{\partial w}{\partial y}n_y\right) + \left(N_{xy}\frac{\partial w}{\partial y}n_x\right) + \left(N_{xy}\frac{\partial w}{\partial x}n_y\right)\right]\delta w \, ds$$
$$+ \int\int \left(N_{xx}\frac{\partial^2 w}{\partial x^2} + N_{yy}\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y}\right) - (Q_{x,x} + Q_{y,y})\,\delta w \, dxdy = 0 \tag{3.25}$$

For this expression to be kinematically admissible each seperate section must equal zero. From the field integrals we get the three partial differential equations, and from the boundary integrals we get the boundary conditions.

We present the system of partial differential equations first.

From the section coupled with $\delta\phi_x$ we get

1.
$$-\left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y}\right) + Q_x = 0 \tag{3.26}$$

From the section coupled with $\delta\phi_y$ we get

2.

$$-\left(\frac{\partial M_{yy}}{\partial y}+\frac{\partial M_{xy}}{\partial x}\right)+Q_y=0 \tag{3.27}$$

From the section coupled with δw we get

3.

$$-\left(\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}\right)+N_x\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}+N_y\frac{\partial^2 w}{\partial y^2}=0 \tag{3.28}$$

The above equations can be expressed in terms of displacements (ϕ_x, ϕ_y and w) by substituting for the force and moment resultants.

1.

$$-\frac{D(1-\nu)}{2}\left(\frac{\partial^2\phi_x}{\partial x^2}+\frac{\partial^2\phi_x}{\partial y^2}\right)-\frac{D(1+\nu)}{2}\frac{\partial}{\partial x}\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial\phi_y}{\partial y}\right)+\frac{\kappa Eh}{2(1+\nu)}\left(\frac{\partial w}{\partial x}+\phi_x\right)=0 \tag{3.29}$$

2.

$$-\frac{D(1-\nu)}{2}\left(\frac{\partial^2\phi_y}{\partial x^2}+\frac{\partial^2\phi_y}{\partial y^2}\right)-\frac{D(1+\nu)}{2}\frac{\partial}{\partial y}\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial\phi_y}{\partial y}\right)+\frac{\kappa Eh}{2(1+\nu)}\left(\frac{\partial w}{\partial y}+\phi_y\right)=0 \tag{3.30}$$

3.

$$-\frac{\kappa Eh}{2(1+\nu)}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial\phi_x}{\partial x}+\frac{\partial\phi_y}{\partial y}+\right)+N_{xx}\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}+N_{yy}\frac{\partial^2 w}{\partial y^2}=0 \tag{3.31}$$

It can be usefull to express the differential equations by the moment sum, also called the Marcus moment [11].

Introducing the moment sum

$$M_s\equiv\frac{M_{xx}+M_{yy}}{1+\nu}=D\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial\phi_y}{\partial y}\right) \tag{3.32}$$

and using the Laplace operator

$$\triangledown^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \tag{3.33}$$

, the differential equations can be expressed in the form

1.

$$-D(1-\nu)\nabla^2\phi_x - (1+\nu)\frac{\partial M_s}{\partial x} + \frac{\kappa Eh}{(1+\nu)}\left(\frac{\partial w}{\partial x} + \phi_x\right) = 0 \tag{3.34}$$

2.

$$-D(1-\nu)\nabla^2\phi_y - (1+\nu)\frac{\partial M_s}{\partial y} + \frac{\kappa Eh}{(1+\nu)}\left(\frac{\partial w}{\partial y} + \phi_y\right) = 0 \tag{3.35}$$

3.

$$-\frac{\kappa Eh}{(1+\nu)}\left(\nabla^2 w + \frac{M_s}{D}\right) = N_{xx}\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_{yy}\frac{\partial^2 w}{\partial y^2} \tag{3.36}$$

3.4 Boundary conditions

The boundary conditions are found from the boundary integrals in Eq. (3.25), and involve specifying one element of the following three pairs. On an edge parallel to the x or y coordinate

$$either \quad \delta\phi_x = 0 \quad or \quad M_{xx}n_x + M_{xy}n_y = 0 \tag{3.37}$$

$$either \quad \delta\phi_y = 0 \quad or \quad M_{yy}n_y + M_{xy}n_x = 0 \tag{3.38}$$

$$\begin{aligned} either \quad \delta w = 0 \quad or \quad & Q_x n_x + Q_y n_y \\ & - \left(N_{xx}\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right) n_x \\ & - \left(N_{yy}\frac{\partial w}{\partial y} + N_{xy}\frac{\partial w}{\partial x}\right) n_y = 0 \end{aligned} \tag{3.39}$$

3.5 Solving the Mindlin buckling problem

We consider a plate under uniform in-plane compressive load on all edges, thus $N_{xx} = N_{yy} = N$ and $N_{xy} = 0$.

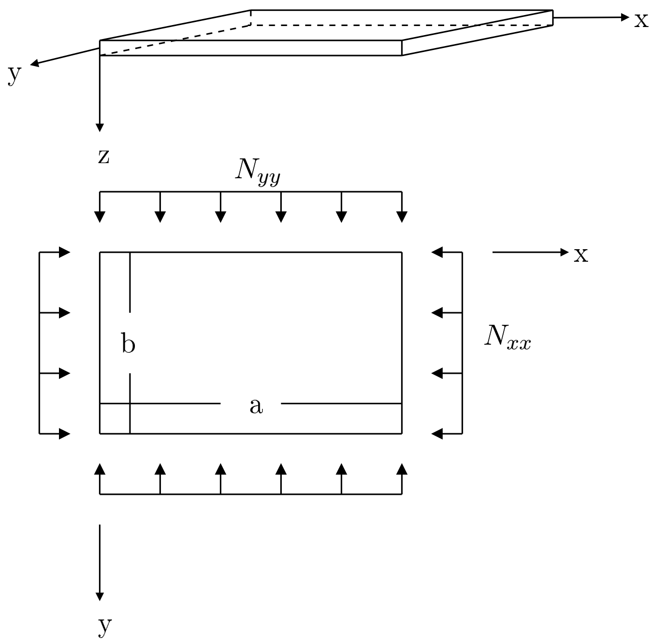

Figure 3.1: Simply supported plate under hydrostatic in-plane load.

The boundary conditions for the simply supported plate are

$$\left.\begin{array}{lll} x = 0: & w(0, y) = 0 & M_{xx}(0, y) = 0 \\ x = a: & w(a, y) = 0 & M_{xx}(a, y) = 0 \\ y = 0: & w(x, 0) = 0 & M_{xx}(x, 0) = 0 \\ y = b: & w(x, b) = 0 & M_{xx}(x, b) = 0 \end{array}\right\} \tag{3.40}$$

We have previously found three equilibrium equations in Eqs. (3.26), (3.27), and (3.28). Note that N_{xy} does not appear in the final equilibrium equation.

From the section coupled with $\delta\phi_x$ we have

1.

$$-\left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y}\right) + Q_x = 0 \tag{3.41}$$

From the section coupled with $\delta\phi_y$ we have

2.

$$-\left(\frac{\partial M_{yy}}{\partial y}+\frac{\partial M_{xy}}{\partial x}\right)+Q_y=0 \tag{3.42}$$

From the section coupled with δw we have

3.

$$-\left(\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}\right)+N_{xx}\frac{\partial^2 w}{\partial x^2}+N_{yy}\frac{\partial^2 w}{\partial y^2}=0 \tag{3.43}$$

The plate constitutive equations are given by

$$M_{xx}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_x z dz=D\left(\frac{\partial \phi_x}{\partial x}+\nu\frac{\partial \phi_y}{\partial y}\right) \tag{3.44a}$$

$$M_{yy}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_y z dz=D\left(\nu\frac{\partial \phi_x}{\partial x}+\frac{\partial \phi_y}{\partial y}\right) \tag{3.44b}$$

$$M_{xy}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{xy} z dz=\frac{D(1-\nu)}{2}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right) \tag{3.44c}$$

$$Q_x=\kappa\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{xz} dz=\frac{\kappa E h}{2(1+\nu)}\left(\phi_x+\frac{\partial w}{\partial x}\right) \tag{3.44d}$$

$$Q_y=\kappa\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{yz} dz=\frac{\kappa E h}{2(1+\nu)}\left(\phi_y+\frac{\partial w}{\partial y}\right) \tag{3.44e}$$

Introducing the constitutive equations into the equations of equlibrium gives

1.

$$-D\left(\frac{\partial^2 \phi_x}{\partial x^2}+\nu\frac{\partial^2 \phi_y}{\partial x\partial y}\right)-\frac{D(1-\nu)}{2}\left(\frac{\partial^2 \phi_x}{\partial y^2}+\frac{\partial^2 \phi_y}{\partial x\partial y}\right)+\frac{\kappa E h}{2(1+\nu)}\left(\phi_x+\frac{\partial w}{\partial x}\right)=0$$

2.

$$-D\left(\frac{\partial^2 \phi_y}{\partial y^2}+\nu\frac{\partial^2 \phi_x}{\partial x\partial y}\right)-\frac{D(1-\nu)}{2}\left(\frac{\partial^2 \phi_y}{\partial x^2}+\frac{\partial^2 \phi_x}{\partial x\partial y}\right)+\frac{\kappa E h}{2(1+\nu)}\left(\phi_y+\frac{\partial w}{\partial y}\right)=0$$

3.

$$-\frac{\kappa E h}{2(1+\nu)}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial \phi_x}{\partial x}+\frac{\partial \phi_y}{\partial y}\right)+N_{xx}\frac{\partial^2 w}{\partial x^2}+N_{yy}\frac{\partial^2 w}{\partial y^2}=0$$

For an isotropic plate we have

$$\left.\begin{aligned} D_{11} &= D \\ D_{12} &= \nu D \\ D_{66} &= \frac{D(1-\nu)}{2} \\ K &= \frac{\kappa E h}{2(1+\nu)} \end{aligned}\right\} \tag{3.45}$$

the equation set that solves the buckling problem is

1.
$$-D_{11}\frac{\partial^2 \phi_x}{\partial x^2} - D_{12}\frac{\partial^2 \phi_y}{\partial x \partial y} - D_{66}\left(\frac{\partial^2 \phi_x}{\partial y^2} + \frac{\partial^2 \phi_y}{\partial x \partial y}\right) + K\left(\phi_x + \frac{\partial w}{\partial x}\right) = 0$$

2.
$$-D_{11}\frac{\partial^2 \phi_y}{\partial y^2} - D_{12}\frac{\partial^2 \phi_x}{\partial x \partial y} - D_{66}\left(\frac{\partial^2 \phi_y}{\partial x^2} + \frac{\partial^2 \phi_x}{\partial x \partial y}\right) + K\left(\phi_y + \frac{\partial w}{\partial y}\right) = 0$$

3.
$$-K\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y}\right) + N_{xx}\frac{\partial^2 w}{\partial x^2} + N_{yy}\frac{\partial^2 w}{\partial y^2} = 0$$

We assume that w, ϕ_x and ϕ_y can be represented by the following double Fourier series

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn} sin\left(\frac{m\pi x}{a}\right) sin\left(\frac{n\pi y}{b}\right) \tag{3.46 a}$$

$$\phi_x(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} X_{mn} cos\left(\frac{m\pi x}{a}\right) sin\left(\frac{n\pi y}{b}\right) \tag{3.46 b}$$

$$\phi_y(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Y_{mn} sin\left(\frac{m\pi x}{a}\right) cos\left(\frac{n\pi y}{b}\right) \tag{3.46 c}$$

where W_{mn}, X_{mn} and Y_{mn} are series coefficients, m and n are positive integers.

Introducing the Fourier series into the equation set gives the following matrix system

$$\begin{bmatrix} -D_{11}\alpha^2 - D_{66}\beta^2 - K & -D_{12}\alpha\beta - D_{66}\alpha\beta & -K\alpha \\ -D_{12}\alpha\beta - D_{66}\alpha\beta & -D_{11}\beta^2 - D_{66}\alpha^2 - K & -K\beta \\ -K\alpha & -K\beta & N\left(\alpha^2+\beta^2\right) - K\alpha^2 - K\beta^2 \end{bmatrix} \begin{Bmatrix} X_{mn} \\ Y_{mn} \\ W_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.47}$$

where $\alpha = \frac{m\pi}{a}$ and $\beta = \frac{n\pi}{b}$.

By defining

$$\left.\begin{aligned}
C_1 &= -D_{11}\alpha^2 - D_{66}\beta^2 - K \\
C_2 &= -D_{12}\alpha\beta - D_{66}\alpha\beta \\
C_3 &= -K\alpha \\
C_4 &= -D_{11}\beta^2 - D_{66}\alpha^2 - K \\
C_5 &= -K\beta \\
C_6 &= N\left(\alpha^2 + \beta^2\right) + \alpha C_3 + \beta C_5
\end{aligned}\right\} \tag{3.48}$$

we simplify the matrix.

$$\begin{bmatrix} C_1 & C_2 & C_3 \\ C_2 & C_4 & C_5 \\ C_3 & C_5 & C_6 \end{bmatrix} \begin{Bmatrix} X_{mn} \\ Y_{mn} \\ W_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.49}$$

The non trivial solution we are seeking is found by setting the determinant of the matrix in Eq. (3.49) equal to zero [13], [14].

$$\begin{vmatrix} C_1 & C_2 & C_3 \\ C_2 & C_4 & C_5 \\ C_3 & C_5 & C_6 \end{vmatrix} = 0 \tag{3.50}$$

$$\begin{aligned}
&C_1C_4C_6 - C_1C_5^2 - C_2^2C_6 + C_2C_3C_5 + C_2C_3C_5 - C_3^2C_4 = 0 \\
&C_6(C_1C_4 - C_2^2) = C_1C_5^2 + C_3^2C_4 - 2C_2C_3C_5
\end{aligned} \tag{3.51}$$

Inserting for C_6 and solving for N gives

$$N = \left(\frac{C_1C_5^2 + C_3^2C_4 - 2C_2C_3C_5}{(C_1C_4 - C_2^2)} - \alpha C_3 - \beta C_5\right)\frac{1}{\alpha^2 + \beta^2} \tag{3.52}$$

The critical buckling load N_{cr} occurs at $n = 1$ while m can vary.

Chapter 4

HSDT; a gradient elasticity approach

4.1 Gradient elasticity FSDT model

The objective is to describe the energy functional ($\Pi = U + V$) of higher order shear plate theories in a gradient elasticity first order shear plate model, meaning that one or more variables in HSDT are gradients of the variables in FSDT. The potential energy of external forces V is the same in HSDT and FSDT. The challenge is therefore describing the strain energy U. We will focus on the theory of Reddy and the one of Shi, which are based on different kinematics functions $f(z)$. We want to show that both can be classified in a common gradient elasticity first order shear plate theory.

The two theories are based on the following kinematics functions:

- Reddy: $f(z) = z\left(1 - \frac{4z^2}{3h^2}\right)$
- Shi: $f(z) = \frac{5}{4}z\left(1 - \frac{4z^2}{3h^2}\right)$

Note that the polynomials are of third order, meaning that the displacements are described by a function of third order. We want to ceep these functions fixed and derive the energy equations without inserting for $f(z)$.

To describe the theories in a unified way we introduce the parameter ζ which takes different value in the various HSDTs, depending on the respective kinematics function. The unified kinematics function can then be presented as

$$f(z) = \zeta z\left(1 - \frac{4z^2}{3h^2}\right) \tag{4.1}$$

where

- $\zeta = 1$ in Reddy's theory
- $\zeta = \frac{5}{4}$ in Shi's theory

The displacement field for a plate in HSDT can be described by:

$$\left.\begin{aligned} u(x,y,z) &= u_0(x,y) - zw_{,x} + f(z)\left(\phi_x + w_{,x}\right) \\ v(x,y,z) &= v_0(x,y) - zw_{,y} + f(z)\left(\phi_y + w_{,y}\right) \\ w(x,y,z) &= w(x,y) \end{aligned}\right\} \tag{4.2}$$

The in-plane displacement of the middle surface (u_0, and v_0) will in the following be omitted without affecting the basic bending features of shear flexible plates.

In view of that displacement field the strains are given by

$$\left.\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} = -zw_{,xx} + f(z)\left(\phi_{x,x} + w_{,xx}\right) = f(z)\phi_{x,x} + \left[f(z) - z\right] w_{,xx} \\ \epsilon_y &= \frac{\partial v}{\partial y} = -zw_{,yy} + f(z)\left(\phi_{y,y} + w_{,yy}\right) = f(z)\phi_{y,y} + \left[f(z) - z\right] w_{,yy} \\ \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = f'(z)\left[\phi_x + w_{,x}\right] \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = f'(z)\left[\phi_y + w_{,y}\right] \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ &= -zw_{,xy} + f(z)\left(\phi_{x,y} + w_{,xy}\right) - zw_{,yx} + f(z)\left(\phi_{y,x} + w_{,yx}\right) \\ &= f(z)\phi_{x,y} + \left[f(z) - z\right] w_{,xy} + f(z)\phi_{y,x} + \left[f(z) - z\right] w_{,yx} \end{aligned}\right\} \tag{4.3}$$

Organizing the above strain expressions to release the gradients of the deflection (w) and rotations (ϕ_x and ϕ_y):

$$\left.\begin{aligned} \epsilon_x &= z\phi_{x,x} + \left(f(z) - z\right)\left(w_{,xx} + \phi_{x,x}\right) \\ \epsilon_y &= z\phi_{y,y} + \left(f(z) - z\right)\left(w_{,yy} + \phi_{y,y}\right) \\ \gamma_{xz} &= f'(z)\left[w_{,x} + \phi_x\right] \\ \gamma_{yz} &= f'(z)\left[w_{,y} + \phi_y\right] \\ \gamma_{xy} &= z\phi_{x,y} + \left(f(z) - z\right)\left(w_{,xy} + \phi_{x,y}\right) \\ &\quad + z\phi_{y,x} + \left(f(z) - z\right)\left(w_{,yx} + \phi_{y,x}\right) \end{aligned}\right\} \tag{4.4}$$

Considering an orthotropic elastic constitutive law:

$$
\left.
\begin{aligned}
\sigma_x &= Q_{11}\epsilon_x + Q_{12}\epsilon_y \\
\sigma_y &= Q_{12}\epsilon_x + Q_{22}\epsilon_y \\
\tau_{xz} &= Q_{44}\gamma_{xz} \\
\tau_{yz} &= Q_{55}\gamma_{yz} \\
\tau_{xy} &= Q_{66}\gamma_{yz}
\end{aligned}
\right\} \tag{4.5}
$$

In matrix form:

$$
\left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xz} \\ \tau_{yz} \\ \tau_{xy} \end{array} \right\} = \left[\begin{array}{ccccc} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{array} \right] \left\{ \begin{array}{c} \epsilon_x \\ \epsilon_y \\ \gamma_{xz} \\ \gamma_{yz} \\ \gamma_{xy} \end{array} \right\} \tag{4.6}
$$

where Q_{ij} are the plane stress reduced elastic constants in the material axes of the plate.

In general the strain energy under the plain stress assumption can be written:

$$
U = \frac{1}{2} \int \int \int (\sigma_x\epsilon_x + \sigma_y\epsilon_y + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz} + \tau_{xy}\gamma_{xy})\, dxdydz \tag{4.7}
$$

See Eqs. (2.6) and (2.8).

Introducing stresses from Eq. (4.6) gives

$$
U = \frac{1}{2} \int \int \int \left(Q_{11}\epsilon_x^2 + 2Q_{12}\epsilon_x\epsilon_y + Q_{22}\epsilon_y^2 + Q_{44}\gamma_{xz}^2 + Q_{55}\gamma_{yz}^2 + Q_{66}\gamma_{xy}^2 \right) dxdydz \tag{4.8}
$$

Itroducing strain expressions from Eq. (4.4) gives

$$
\begin{aligned}
U = \frac{1}{2}\int\int\int\Big[& Q_{11}\left[z\phi_{x,x} + (f(z)-z)\left(w_{,xx}+\phi_{x,x}\right)\right]^2 \\
& +2Q_{12}\left[z\phi_{x,x} + (f(z)-z)\left(w_{,xx}+\phi_{x,x}\right)\right]\left[z\phi_{y,y} + (f(z)-z)\left(w_{,yy}+\phi_{y,y}\right)\right] \\
& +Q_{22}\left[z\phi_{y,y} + (f(z)-z)\left(w_{,yy}+\phi_{y,y}\right)\right]^2 \\
& +Q_{44}\left[f'(z)\left[w_{,x}+\phi_{x}\right]\right]^2 \\
& +Q_{55}\left[f'(z)\left[w_{,y}+\phi_{y}\right]\right]^2 \\
& +Q_{66}\left[\phi_{x,y} + (f(z)-z)\left(w_{,xy}+\phi_{x,y}\right) + z\phi_{y,x} + (f(z)-z)\left(w_{,yx}+\phi_{y,x}\right)\right]^2\Big]dxdydz
\end{aligned} \tag{4.9}
$$

We need to identify the variables we are delaing with in Eq. (4.9). If we enumerate the six lines in Eq. (4.9) from 1-6 and treat them separately we have

1.

$$
U = \frac{1}{2}\int\int\int Q_{11}\left[z\phi_{x,x} + (f(z)-z)\left(w_{,xx}+\phi_{x,x}\right)\right]^2 dV
$$

$$
U = Q_{11}\frac{1}{2}\int\int\int\left[z^2\phi_{x,x}^2 + 2z\phi_{x,x}\left(f(z)-z\right)\left(w_{,xx}+\phi_{x,x}\right) + \left(f(z)-z\right)^2\left(w_{,xx}+\phi_{x,x}+\right)^2\right]dV
$$

From 1. we recognize the two variables: $\phi_{x,x}$ and $(w_{,xx}+\phi_{x,x})$.

2.

$$
U = \frac{1}{2}\int\int\int 2Q_{12}\left[z\phi_{x,x} + (f(z)-z)\left(w_{,xx}+\phi_{x,x}\right)\right]\left[z\phi_{y,y} + (f(z)-z)\left(w_{,yy}+\phi_{y,y}\right)\right]dV
$$

$$
\begin{aligned}
U = 2Q_{12}\frac{1}{2}\int\int\int\Big[& z^2\phi_{x,x}\phi_{y,y} + z\left(f(z)-z\right)\phi_{x,x}\left(w_{,yy}+\phi_{y,y}\right) + z\left(f(z)-z\right)\phi_{y,y}\left(w_{,xx}+\phi_{x,x}\right) \\
& + \left(f(z)-z\right)^2\left(w_{,xx}+\phi_{x,x}\right)\left(w_{,yy}+\phi_{y,y}\right)\Big]dV
\end{aligned}
$$

From 2. we recognize the four variables: $\phi_{x,x}$, $\phi_{y,y}$, $(w_{,xx}+\phi_{x,x})$ and $(w_{,yy}+\phi_{y,y})$, which are the same as in 1. and 3.

3.

$$
U = \frac{1}{2}\int\int\int Q_{22}\left[z\phi_{y,y} + (f(z)-z)\left(w_{,yy}+\phi_{y,y}\right)\right]^2 dV
$$

$$
U = Q_{22}\frac{1}{2}\int\int\int\left[z^2\phi_{y,y}^2 + 2z\phi_{y,y}\left(f(z)-z\right)\left(w_{,yy}+\phi_{y,y}\right) + \left(f(z)-z\right)^2\left(w_{,yy}+\phi_{y,y}+\right)^2\right]dV
$$

From 3. we recognize the two variables: $\phi_{y,y}$ and $(w_{,yy}+\phi_{y,y})$.

4.

$$
U = \frac{1}{2}\int\int\int Q_{44}\left[f'(z)\left(w_{,x}+\phi_{x}\right)\right]^2 dV
$$

$$
U = Q_{44}\frac{1}{2}\int\int\int\left[f'(z)^2\left(w_{,x}+\phi_{x}\right)^2\right]dV
$$

From 4. we recognize the one variable: $(w_{,x}+\phi_{x})$.

5.

$$U = \frac{1}{2}\int\int\int Q_{55}\left[f'(z)\left(w_{,y}+\phi_y\right)\right]^2 dV$$
$$U = Q_{55}\frac{1}{2}\int\int\int\left[f'(z)^2\left(w_{,y}+\phi_y\right)^2\right] dV$$

From 5. we recognize the one variable: $(w_{,y}+\phi_y)$.

6.

$$U = \frac{1}{2}\int\int\int\left[Q_{66}\left[z\phi_{x,y}+(f(z)-z)\left(w_{,xy}+\phi_{x,y}\right)+z\phi_{y,x}+(f(z)-z)\left(w_{,xy}+\phi_{y,x}\right)\right]^2\right]dV$$
$$U = Q_{66}\frac{1}{2}\int\int\int\Big[$$
$$z^2\phi_{x,y}^2+z\left(f(z)-z\right)\phi_{x,y}\left(w_{,xy}+\phi_{x,y}\right)+z^2\phi_{x,y}\phi_{y,x}+z\left(f(z)-z\right)\phi_{x,y}\left(w_{,xy}+\phi_{y,x}\right)$$

$$+z\left(f(z)-z\right)\phi_{x,y}\left(w_{,xy}+\phi_{x,y}\right)+\left(f(z)-z\right)^2\left(w_{,xy}+\phi_{x,y}\right)^2$$
$$+z\left(f(z)-z\right)\phi_{y,x}\left(w_{,xy}+\phi_{x,y}\right)+\left(f(z)-z\right)^2\left(w_{,xy}+\phi_{x,y}\right)\left(w_{,xy}+\phi_{y,x}\right)$$

$$+z^2\phi_{y,x}\phi_{x,y}+z\left(f(z)-z\right)\phi_{y,x}\left(w_{,xy}+\phi_{x,y}\right)+z^2\phi_{y,x}^2+z\left(f(z)-z\right)\phi_{y,x}\left(w_{,xy}+\phi_{y,x}\right)$$

$$+z\left(f(z)-z\right)\phi_{y,x}\left(w_{,xy}+\phi_{y,x}\right)+\left(f(z)-z\right)^2\left(w_{,xy}+\phi_{y,x}\right)\left(w_{,xy}+\phi_{x,y}\right)$$
$$+z\left(f(z)-z\right)\phi_{y,x}\left(w_{,xy}+\phi_{y,x}\right)+\left(f(z)-z\right)^2\left(w_{,xy}+\phi_{y,x}\right)^2\Big]dV$$

From 6. we recognize the four variables: $\phi_{x,y}$, $\phi_{y,x}$, $(w_{,xy}+\phi_{x,y})$ and $(w_{,xy}+\phi_{y,x})$.

We have now targeted the ten variables:

$$\left\{\phi_{x,x}\quad \phi_{y,y}\quad \phi_{x,y}\quad \phi_{y,x}\quad (w_{,x}+\phi_x)\ (w_{,y}+\phi_y)\ (w_{,xx}+\phi_{x,x})\ (w_{,yy}+\phi_{y,y})\ (w_{,xy}+\phi_{x,y})\ (w_{,xy}+\phi_{y,x})\right\}$$

The next thing we want to do is to write the strain energy including the above lines 1-6 in a 10x10 matrix system. We will show that the 10x10 stiffness matrix can be reduced to an 8x8 matrix. The stiffness parameters that enter the matrix system can be introduced in the following format:

$$\int_{-h/2}^{h/2} Q_{ij} z^2 dz = D_{ij} \tag{4.10a}$$

$$\int_{-h/2}^{h/2} Q_{ii} \left[f'(z)\right]^2 dz = \kappa A_{ii} \tag{4.10b}$$

$$\int_{-h/2}^{h/2} Q_{ij} \left[f(z) - z\right]^2 dz = b_0^2 \kappa A_{ij} \tag{4.10c}$$

$$\int_{-h/2}^{h/2} Q_{ij} z \left[f(z) - z\right] dz = -c_0^2 \kappa A_{ij} \tag{4.10d}$$

where $i, j = 1, 2, 6$ and $i, i = 4, 5$. κ in HSDT is not a correction factor that we assume to be for example $\kappa = \frac{5}{6}$ as in FSDT. In HSDT κ is a constant computed from the kinematics.

We define

$$T_{ij} = \kappa A_{ij} \quad and \quad T_{ii} = \kappa A_{ii} \tag{4.11}$$

so that D_{ij} is the flexural stiffness and T_{ij} and T_{ii} is the transverse shear stiffness of an orthotropic plate. We will return to the stiffness parameters in Section 4.2 where they are further discussed.

The strain energy is now written in matrix form as:

$$U = \frac{1}{2}\int\int$$

$$\left\{\phi_{x,x} \quad \phi_{y,y} \quad \phi_{x,y} \quad \phi_{y,x} \quad (w_{,x}+\phi_x) \quad (w_{,y}+\phi_y) \quad (w_{,xx}+\phi_{x,x}) \quad (w_{,yy}+\phi_{y,y}) \quad (w_{,xy}+\phi_{x,y}) \quad (w_{,xy}+\phi_{y,x})\right\}\cdot$$

$$\begin{bmatrix} D_{11} & D_{12} & 0 & 0 & 0 & 0 & -c_0^2T_{11} & -c_0^2T_{12} & 0 & 0 \\ D_{12} & D_{22} & 0 & 0 & 0 & 0 & -c_0^2T_{12} & -c_0^2T_{22} & 0 & 0 \\ 0 & 0 & D_{66} & D_{66} & 0 & 0 & 0 & 0 & -c_0^2T_{66} & -c_0^2T_{66} \\ 0 & 0 & D_{66} & D_{66} & 0 & 0 & 0 & 0 & -c_0^2T_{66} & -c_0^2T_{66} \\ 0 & 0 & 0 & 0 & T_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & T_{55} & 0 & 0 & 0 & 0 \\ -c_0^2T_{11} & -c_0^2T_{12} & 0 & 0 & 0 & 0 & b_0^2T_{11} & b_0^2T_{12} & 0 & 0 \\ -c_0^2T_{12} & -c_0^2T_{22} & 0 & 0 & 0 & 0 & b_0^2T_{12} & b_0^2T_{22} & 0 & 0 \\ 0 & 0 & -c_0^2T_{66} & -c_0^2T_{66} & 0 & 0 & 0 & 0 & b_0^2T_{66} & b_0^2T_{66} \\ 0 & 0 & -c_0^2T_{66} & -c_0^2T_{66} & 0 & 0 & 0 & 0 & b_0^2T_{66} & b_0^2T_{66} \end{bmatrix}\cdot$$

$$\begin{Bmatrix} \phi_{x,x} \\ \phi_{y,y} \\ \phi_{x,y} \\ \phi_{y,x} \\ (w_{,x}+\phi_x) \\ (w_{,y}+\phi_y) \\ (w_{,xx}+\phi_{x,x}) \\ (w_{,yy}+\phi_{y,y}) \\ (w_{,xy}+\phi_{x,y}) \\ (w_{,xy}+\phi_{y,x}) \end{Bmatrix} dxdy \tag{4.12}$$

We observe in the 10x10 stiffness matrix that rows and columns 3 and 4 (connected to the variables ($\phi_{x,y}$ and $\phi_{y,x}$) are identical. Hence, the two variables can be combined to the one variable ($\phi_{x,y}+\phi_{y,x}$), and the rows and columns are merged. The same goes for rows and columns 9 and 10 (connected to the variables $(w_{,xy}+\phi_{x,y})$ and $(w_{,xy}+\phi_{y,x})$). The two variables are summed to one variable, namely $(\phi_{x,y}+2w_{,xy}+\phi_{y,x})$, and the rows and columns are merged.

The matrix in Eq. (4.12) is now reduced to the 8x8 stiffness matrix:

$$U = \frac{1}{2} \int \int$$

$$\Big\{ \phi_{x,x} \;\; \phi_{y,y} \;\; (\phi_{x,y} + \phi_{y,x}) \;\; (w_{,x} + \phi_x) \;\; (w_{,y} + \phi_y) \;\; (w_{,xx} + \phi_{x,x}) \;\; (w_{,yy} + \phi_{y,y}) \;\; (\phi_{x,y} + 2w_{,xy} + \phi_{y,x}) \Big\} \cdot$$

$$\begin{bmatrix} D_{11} & D_{12} & 0 & 0 & 0 & -c_0^2 T_{11} & -c_0^2 T_{12} & 0 \\ D_{12} & D_{22} & 0 & 0 & 0 & -c_0^2 T_{12} & -c_0^2 T_{22} & 0 \\ 0 & 0 & D_{66} & 0 & 0 & 0 & 0 & -c_0^2 T_{66} \\ 0 & 0 & 0 & T_{44} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & T_{55} & 0 & 0 & 0 \\ -c_0^2 T_{11} & -c_0^2 T_{12} & 0 & 0 & 0 & b_0^2 T_{11} & b_0^2 T_{12} & 0 \\ -c_0^2 T_{12} & -c_0^2 T_{22} & 0 & 0 & 0 & b_0^2 T_{12} & b_0^2 T_{22} & 0 \\ 0 & 0 & -c_0^2 T_{66} & 0 & 0 & 0 & 0 & b_0^2 T_{66} \end{bmatrix} \cdot$$

$$\begin{Bmatrix} \phi_{x,x} \\ \phi_{y,y} \\ (\phi_{x,y} + \phi_{y,x}) \\ (w_{,x} + \phi_x) \\ (w_{,y} + \phi_y) \\ (w_{,xx} + \phi_{x,x}) \\ (w_{,yy} + \phi_{y,y}) \\ (\phi_{x,y} + 2w_{,xy} + \phi_{y,x}) \end{Bmatrix} dxdy \tag{4.13}$$

From this matrix system we make several important observations. First the matrix is symmetric, as it should be. Second we notice that rows and columns 6, 7 and 8 include two additional length scales, namely b_0 and c_0. If we omit rows and columns 6, 7 and 8 we get the same matrix system as in FSDT, see Eq. (3.10). We also notice that the three variables connected to rows and columns 6, 7 and 8 are the gradients of the variables connected to rows and columns 4 and 5. This leads to the conclusion that we are dealing with a gradient elasticity FSDT model.

4.2 Stiffness parameters

The stiffness parameters in Eq. (4.10), repeated here for ready reference, are calculated for each HSDT as follows. We will utilize the unified kinematics function in Eq. (4.1) to obtain dimensionless values for the stiffness parameters. We will see that the length scale c_0 will vanish in the case of Shi. The dimensionless value of the stiffness parameters play a key role in describing the two HSDTs in a unified framework, and they are presnted for Shi's and Reddy's theories in Table 4.1.

$$\int_{-h/2}^{h/2} Q_{ij} z^2 dz = D_{ij} \tag{4.14a}$$

$$\int_{-h/2}^{h/2} Q_{ii} \left[f'(z)\right]^2 dz = \kappa A_{ii} \tag{4.14b}$$

$$\int_{-h/2}^{h/2} Q_{ij} \left[f(z) - z\right]^2 dz = b_0^2 \kappa A_{ij} \tag{4.14c}$$

$$\int_{-h/2}^{h/2} Q_{ij} z \left[f(z) - z\right] dz = -c_0^2 \kappa A_{ij} \tag{4.14d}$$

where $i, j = 1, 2, 6$, $i, i = 4, 5$, $\kappa A_{ij} = T_{ij}$ and $\kappa A_{ii} = T_{ii}$.

The unified kinematics function is

$$f(z) = \zeta z \left(1 - \frac{4z^2}{3h^2}\right) \tag{4.15}$$

The derivative of $f(z)$ is

$$f'(z) = \zeta - \zeta \frac{4z^2}{h^2} \tag{4.16}$$

Recall that ζ takes the value

- 1 for Reddy
- $\frac{5}{4}$ for Shi

$$\int_{-h/2}^{h/2} Q_{ij} z^2 dz = Q_{ij} \frac{h^3}{12} = D_{ij} \tag{4.17}$$

$$
\begin{aligned}
&\int_{-h/2}^{h/2} Q_{ii}\left[f'(z)\right]^2 dz = T_{ii} \\
&\int_{-h/2}^{h/2} Q_{ii}\left[\zeta - \zeta\frac{4z^2}{h^2}\right]^2 dz = T_{ii} \\
&\int_{-h/2}^{h/2} Q_{ii}\left(\zeta^2 - \zeta^2\frac{8z^2}{h^2} + \zeta^2\frac{16z^4}{h^4}\right) dz = T_{ii} \\
&Q_{ii}\left[\zeta^2 z - \zeta^2\frac{8z^3}{3h^2} + \zeta^2\frac{16z^5}{5h^4}\right]_{-h/2}^{h/2} = T_{ii} \\
&Q_{ii}\left[2\zeta^2\frac{h}{2} - \zeta^2\frac{16}{3h^2}\frac{h^3}{8} + \zeta^2\frac{32}{5h^4}\frac{h^5}{32}\right] = T_{ii} \\
&Q_{ii}\zeta^2 h\left[1 - \frac{2}{3} + \frac{1}{5}\right] = T_{ii} \\
&\underline{\underline{\frac{T_{ii}}{Q_{ii}h} = \zeta^2\frac{8}{15}}}
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
&\int_{-h/2}^{h/2} Q_{ij}\left[f(z) - z\right]^2 dz = b_0^2 T_{ij} \\
&\int_{-h/2}^{h/2} Q_{ij}\left(\zeta z - \zeta\frac{4z^3}{3h^2} - z\right)^2 dz = b_0^2 T_{ij} \\
&\int_{-h/2}^{h/2} Q_{ij}\left(\zeta^2 z^2 + \zeta^2\frac{16z^6}{9h^4} + z^2 - \zeta^2\frac{8z^4}{3h^2} - 2\zeta z^2 + \zeta\frac{8z^4}{3h^2}\right) dz = b_0^2 T_{ij} \\
&Q_{ij}\left[\zeta^2\frac{z^3}{3} + \zeta^2\frac{16}{9h^4}\frac{z^7}{7} + \frac{z^3}{3} - \zeta^2\frac{8}{3h^2}\frac{z^5}{5} - 2\zeta\frac{z^3}{3} + \zeta\frac{8}{3h^2}\frac{z^5}{5}\right]_{-h/2}^{h/2} = b_0^2 T_{ij} \\
&Q_{ij}\left[\zeta^2\frac{2}{3}\frac{h^3}{8} + \zeta^2\frac{32}{63h^4}\frac{h^7}{128} + \frac{2}{3}\frac{h^3}{8} - \zeta^2\frac{16}{15h^2}\frac{h^5}{32} - \zeta\frac{4}{3}\frac{h^3}{8} + \zeta\frac{16}{15h^2}\frac{h^5}{32}\right] = b_0^2 T_{ij} \\
&Q_{ij}h^3\zeta\left[\zeta\frac{1}{12} + \zeta\frac{1}{252} + \frac{1}{12\zeta} - \zeta\frac{1}{30} - \frac{1}{6} + \frac{1}{30}\right] = b_0^2 T_{ij} \\
&Q_{ij}h^3\zeta\left[\zeta\frac{17}{315} + \frac{1}{12\zeta} - \frac{2}{15}\right] = b_0^2 T_{ij} \\
&\underline{\underline{\frac{b_0^2 T_{ij}}{D_{ij}} = \zeta^2\frac{68}{105} - \zeta\frac{8}{5} + 1}}
\end{aligned}
\tag{4.19}
$$

, since $Q_{ij}h^3 = \int_{-h/2}^{h/2} Q_{ij}z^2 \cdot 12dz = 12D_{ij}$, see Eqs. (4.17).

$$
\begin{aligned}
&\int_{-h/2}^{h/2} Q_{ij} z\,[f(z) - z]\,dz = -c_0^2 T_{ij} \\
&\int_{-h/2}^{h/2} Q_{ij} z \left(\zeta z - \zeta \frac{4z^3}{3h^2} - z\right) dz = -c_0^2 T_{ij} \\
&\int_{-h/2}^{h/2} Q_{ij} \left(\zeta z^2 - \zeta \frac{4z^4}{3h^2} - z^2\right) dz = -c_0^2 T_{ij} \\
&Q_{ij} \left[\zeta \frac{z^3}{3} - \zeta \frac{4}{3h^2} \frac{z^5}{5} - \frac{z^3}{3}\right]_{-h/2}^{h/2} = -c_0^2 T_{ij} \\
&Q_{ij} \left[\zeta \frac{2}{3} \frac{h^3}{8} - \zeta \frac{8}{15h^2} \frac{h^5}{32} - \frac{2}{3} \frac{h^3}{8}\right] = -c_0^2 T_{ij} \\
&Q_{ij} h^3 \left[\zeta \frac{1}{12} - \zeta \frac{1}{60} - \frac{1}{12}\right] = -c_0^2 T_{ij} \\
&\underline{\underline{\frac{c_0^2 T_{ij}}{D_{ij}} = 1 - \zeta \frac{4}{5}}}
\end{aligned}
\tag{4.20}
$$

By introducing $\zeta = 1$ for Reddy and $\zeta = \frac{5}{4}$ for Shi, we obtain the values given in Table 4.1.

Table 4.1: Dimensionless stiffness parameters of Shi's and Reddy's HSDT.

HSDT	$f(z)$	$\frac{T_{ii}}{Q_{ii}h}$	$\frac{b_0^2 T_{ij}}{D_{ij}}$	$\frac{c_0^2 T_{ij}}{D_{ij}}$
Shi	$f(z) = \frac{5}{4}z\left(1 - \frac{4z^2}{3h^2}\right)$	$\frac{5}{6}$	$\frac{1}{84}$	0
Reddy	$f(z) = z\left(1 - \frac{4z^2}{3h^2}\right)$	$\frac{8}{15}$	$\frac{1}{21}$	$\frac{1}{5}$

In a similar manner Challamel et al. [5] found the values of the stiffness parameters in the HSDTs of Touratier, Karama et al. and Mechab, shown in Table 4.2.

Table 4.2: Dimensionless stiffness parameters of Touratier's, Karama et al.'s and Mechab's HSDT.

HSDT	$f(z)$	$\frac{T_{ii}}{Q_{ii}h}$	$\frac{b_0^2 T_{ij}}{D_{ij}}$	$\frac{c_0^2 T_{ij}}{D_{ij}}$
Touratier	$f(z) = \frac{h}{\pi} sin\left(\pi \frac{z}{h}\right)$	$\frac{1}{2} = 0,5$	$\frac{5873}{98123} \approx 0,0559$	$\frac{35208}{155813} \approx 0,226$
Karama et al	$f(z) = ze^{-2\left(\frac{z}{h}\right)^2}$	$\frac{16822}{35935} \approx 0,468$	$\frac{9474}{128279} \approx 0,0739$	$\frac{25283}{100044} \approx 0,253$
Mechab	$f(z) = \frac{zcosh\left(\frac{\pi}{2}\right) - \frac{h}{\pi} sinh\left(\pi \frac{z}{h}\right)}{cosh\left(\frac{\pi}{2}\right) - 1}$	$\frac{21999}{39088} \approx 0,536$	$\frac{12007}{311296} \approx 0,0386$	$\frac{5380}{30127} \approx 0,179$

Chapter 5

HSDT

For a plate with transverse shear deformations the principle of minimum potential energy states

$$\delta\Pi = \delta[U(\phi_x, \phi_y, w) + V(w)] = 0 \tag{5.1}$$

We want to use the matrix system in Eq. (4.13) to find an expression for δU and substitute that expression into Eq. (5.1). We also need an expression for δV, but as previously mentioned δV is the same in HSDT as in Mindlin plate theory. In Chapter 4.2 we found

$$\begin{aligned}\delta V = &\int\int \left(N_{xx}\frac{\partial^2 w}{\partial x^2} + N_{yy}\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} \right) \delta w dx dy \\ &- \oint_\Gamma \left[\left(N_{xx}\frac{\partial w}{\partial x}n_x \right) + \left(N_{yy}\frac{\partial w}{\partial y}n_y \right) + \left(N_{xy}\frac{\partial w}{\partial y}n_x \right) + \left(N_{xy}\frac{\partial w}{\partial x}n_y \right) \right] \delta w ds\end{aligned} \tag{5.2}$$

δV is added to δU in Eq. (5.9) below.

5.1 Strain energy

When multiplying out the matrix system in Eq. (4.13) we obtain the following expression for the strain energy:

$$
\begin{aligned}
U = \frac{1}{2} \int \int \Big[\\
& D_{11}\phi_{x,x}^2 + 2D_{12}\phi_{x,x}\phi_{y,y} + D_{22}\phi_{y,y}^2 + D_{66}\left(\phi_{x,y}^2 + 2\phi_{x,y}\phi_{y,x} + \phi_{y,x}^2\right) \\
& + T_{44}\left(w_{,x}^2 + 2w_{,x}\phi_x + \phi_x^2\right) + T_{55}\left(w_{,y}^2 + 2w_{,y}\phi_y + \phi_y^2\right) \\
& + b_0^2 T_{11}\left(w_{,xx}^2 + 2w_{,xx}\phi_{x,x} + \phi_{x,x}^2\right) + b_0^2 T_{22}\left(w_{,yy}^2 + 2w_{,yy}\phi_{y,y} + \phi_{yy}^2\right) \\
& + 2b_0^2 T_{12}\left(w_{,xx}w_{,yy} + w_{,xx}\phi_{y,y} + w_{,yy}\phi_{x,x} + \phi_{x,x}\phi_{y,y}\right) \\
& + b_0^2 T_{66}\left(\phi_{x,y}^2 + 4w_{,xy}^2 + \phi_{y,x}^2 + 4w_{,xy}\phi_{x,y} + 2\phi_{x,y}\phi_{y,x} + 4w_{,xy}\phi_{y,x}\right) \\
& - 2c_0^2 T_{11}\phi_{x,x}\left(w_{,xx} + \phi_{x,x}\right) - 2c_0^2 T_{12}\phi_{x,x}\left(w_{,yy} + \phi_{y,y}\right) \\
& - 2c_0^2 T_{12}\phi_{y,y}\left(w_{,xx} + \phi_{x,x}\right) - 2c_0^2 T_{22}\phi_{y,y}\left(w_{,yy} + \phi_{y,y}\right) \\
& - 2c_0^2 T_{66}\left(\phi_{x,y} + \phi_{y,x}\right)\left(\phi_{x,y} + 2w_{,xy} + \phi_{y,x}\right)\Big] dxdy
\end{aligned}
\tag{5.3}
$$

When using the fundamental lemma of variational calculus we obtain

$$
\begin{aligned}
\delta U = \int\int \Bigg\{ & \\
& \Big[D_{11}\phi_{x,x} + D_{12}\phi_{y,y} + b_0^2 T_{11} w_{,xx} + b_0^2 T_{11}\phi_{x,x} + b_0^2 T_{12} w_{,yy} + b_0^2 T_{12}\phi_{y,y} \\
& - c_0^2 T_{11} w_{,xx} - 2c_0^2 T_{11}\phi_{x,x} - c_0^2 T_{12} w_{,yy} - 2c_0^2 T_{12}\phi_{y,y}\Big]\delta\frac{\partial\phi_x}{\partial x} \\
& + \Big[D_{66}\phi_{x,y} + D_{66}\phi_{y,x} + b_0^2 T_{66}\phi_{x,y} + 2b_0^2 T_{66} w_{,xy} + b_0^2 T_{66}\phi_{y,x} \\
& - 2c_0^2 T_{66}\phi_{x,y} - 2c_0^2 T_{66} w_{,xy} - 2c_0^2 T_{66}\phi_{y,x}\Big]\delta\frac{\partial\phi_x}{\partial y} \\
& + \left[T_{44}\left(w_{,x} + \phi_x\right)\right]\delta\phi_x \\
& + \Big[D_{22}\phi_{y,y} + D_{12}\phi_{x,x} + b_0^2 T_{22} w_{,yy} + b_0^2 T_{22}\phi_{y,y} + b_0^2 T_{12} w_{,xx} + b_0^2 T_{12}\phi_{x,x} \\
& - c_0^2 T_{22} w_{,yy} - 2c_0^2 T_{22}\phi_{y,y} - c_0^2 T_{12} w_{,xx} - 2c_0^2 T_{12}\phi_{x,x}\Big]\delta\frac{\partial\phi_y}{\partial y} \\
& + \Big[D_{66}\phi_{x,y} + D_{66}\phi_{y,x} + b_0^2 T_{66}\phi_{y,x} + 2b_0^2 T_{66} w_{,xy} + b_0^2 T_{66}\phi_{x,y} \\
& - 2c_0^2 T_{66}\phi_{y,x} - 2c_0^2 T_{66} w_{,xy} - 2c_0^2 T_{66}\phi_{x,y}\Big]\delta\frac{\partial\phi_y}{\partial x} \\
& + \left[T_{55}\left(w_{,y} + \phi_y\right)\right]\delta\phi_y \\
& + \Big[T_{44}\left(w_{,x} + \phi_x\right)\Big]\delta\frac{\partial w}{\partial x} + \Big[T_{55}\left(w_{,y} + \phi_y\right)\Big]\delta\frac{\partial w}{\partial y} \\
& + \Big[b_0^2 T_{11} w_{,xx} + b_0^2 T_{11}\phi_{x,x} + b_0^2 T_{12} w_{,yy} + b_0^2 T_{12}\phi_{y,y} - c_0^2 T_{11}\phi_{x,x} - c_0^2 T_{12}\phi_{y,y}\Big]\delta\frac{\partial^2 w}{\partial x^2} \\
& + \Big[b_0^2 T_{22} w_{,yy} + b_0^2 T_{22}\phi_{y,y} + b_0^2 T_{12} w_{,xx} + b_0^2 T_{12}\phi_{x,x} - c_0^2 T_{22}\phi_{y,y} - c_0^2 T_{12}\phi_{x,x}\Big]\delta\frac{\partial^2 w}{\partial y^2} \\
& + \Big[4b_0^2 T_{66} w_{,xy} + 2b_0^2 T_{66}\phi_{x,y} + 2b_0^2 T_{66}\phi_{y,x} - 2c_0^2 T_{66}\phi_{x,y} - 2c_0^2 T_{66}\phi_{y,x}\Big]\delta\frac{\partial^2 w}{\partial x \partial y}\Bigg\} dxdy
\end{aligned}
\tag{5.4}
$$

We need to integrate Eq. (5.4) by parts and include δV to obtain the differential equations and boundary conditions.

The integration process, using the divergence theorem, is executed in the following way, taking the first part of Eq. (5.4) as an example:

$$
\begin{aligned}
&\int\int D_{11} \frac{\partial \phi_x}{\partial x} \delta \frac{\partial \phi_x}{\partial x} \; dxdy \\
&= \oint_{\Gamma} \left(D_{11} \frac{\partial \phi_x}{\partial x} n_x \right) \delta\phi_x \; ds - \frac{\partial}{\partial x} \int\int D_{11} \frac{\partial \phi_x}{\partial x} \delta\phi_x \; dxdy
\end{aligned}
\tag{5.5}
$$

When doing this procedure on all the parts of Eq. (5.4) we obtain a large expression including a field integral and a boundary integral. As we can see from the example in Eq. (5.5) the field integral, and hence the differential equations, is easy to spot from Eq. (5.4). The boundary integral is more complicated, and to abbreviate the terms including b_0 and c_0 we introduce the dimensionless stiffness parameters from Table 4.1 as:

$$\frac{b_0^2 T_{ij}}{D_{ij}} = S_1 \tag{5.6a}$$

$$\frac{c_0^2 T_{ij}}{D_{ij}} = S_2 \tag{5.6b}$$

$$\frac{T_{ii}}{Q_{ii} h} = S_3 \tag{5.6c}$$

so that for example

$$b_0^2 T_{11} \frac{\partial \phi_x}{\partial x} = \frac{b_0^2 T_{11}}{D_{11}} D_{11} \frac{\partial \phi_x}{\partial x} = S_1 D_{11} \frac{\partial \phi_x}{\partial x} \tag{5.7}$$

and

$$T_{44} \left(\frac{\partial w}{\partial x} + \phi_x \right) = \frac{T_{44}}{Q_{44} h} Q_{44} h \left(\frac{\partial w}{\partial x} + \phi_x \right) = S_3 G h \left(\frac{\partial w}{\partial x} + \phi_x \right) \tag{5.8}$$

,since $Q_{44} = Q_{55} = G$.

When integrating Eq. (5.4) and including δV we obtain the expression shown on the next two pages:

$$
\begin{aligned}
&\delta\Pi = -\int\!\!\int \Bigg\{ \\
&\Bigg\{ \frac{\partial}{\partial x}\Bigg[\left(D_{11}\frac{\partial \phi_x}{\partial x} + D_{12}\frac{\partial \phi_y}{\partial y}\right) + b_0^2\left(T_{11}\frac{\partial \phi_x}{\partial x} + T_{12}\frac{\partial \phi_y}{\partial y}\right) + b_0^2\left(T_{11}\frac{\partial^2 w}{\partial x^2} + T_{12}\frac{\partial^2 w}{\partial y^2}\right) \\
&-c_0^2\left(T_{11}\frac{\partial^2 w}{\partial x^2} + T_{12}\frac{\partial^2 w}{\partial y^2}\right) - 2c_0^2\left(T_{11}\frac{\partial \phi_x}{\partial x} + T_{12}\frac{\partial \phi_y}{\partial y}\right)\Bigg] \\
&+\frac{\partial}{\partial y}\left[D_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) + b_0^2 T_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y}\right) - 2c_0^2 T_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + \frac{\partial^2 w}{\partial x \partial y}\right)\right] \\
&-T_{44}\left(\frac{\partial w}{\partial x} + \phi_x\right)\Bigg\}\delta\phi_x \\
&+\Bigg\{ \frac{\partial}{\partial y}\Bigg[\left(D_{22}\frac{\partial \phi_y}{\partial y} + D_{12}\frac{\partial \phi_x}{\partial x}\right) + b_0^2\left(T_{22}\frac{\partial \phi_y}{\partial y} + T_{12}\frac{\partial \phi_x}{\partial x}\right) + b_0^2\left(T_{22}\frac{\partial^2 w}{\partial y^2} + T_{12}\frac{\partial^2 w}{\partial x^2}\right) \\
&-c_0^2\left(T_{12}\frac{\partial^2 w}{\partial x^2} + T_{22}\frac{\partial^2 w}{\partial y^2}\right) - 2c_0^2\left(T_{12}\frac{\partial \phi_x}{\partial x} + T_{22}\frac{\partial \phi_y}{\partial y}\right)\Bigg] \\
&+\frac{\partial}{\partial x}\left[D_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) + b_0^2 T_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y}\right) - 2c_0^2 T_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + \frac{\partial^2 w}{\partial x \partial y}\right)\right] \\
&-T_{55}\left(\frac{\partial w}{\partial y} + \phi_y\right)\Bigg\}\delta\phi_y \\
&+\Bigg\{ \frac{\partial}{\partial x}T_{44}\left(\frac{\partial w}{\partial x} + \phi_x\right) + \frac{\partial}{\partial y}T_{55}\left(\frac{\partial w}{\partial y} + \phi_y\right) \\
&-\frac{\partial^2}{\partial x^2}\left[b_0^2\left(T_{11}\frac{\partial \phi_x}{\partial x} + T_{12}\frac{\partial \phi_y}{\partial y}\right) + b_0^2\left(T_{11}\frac{\partial^2 w}{\partial x^2} + T_{12}\frac{\partial^2 w}{\partial y^2}\right) - c_0^2\left(T_{11}\frac{\partial \phi_x}{\partial x} + T_{12}\frac{\partial \phi_y}{\partial y}\right)\right] \\
&-\frac{\partial^2}{\partial y^2}\left[b_0^2\left(T_{12}\frac{\partial \phi_x}{\partial x} + T_{22}\frac{\partial \phi_y}{\partial y}\right) + b_0^2\left(T_{12}\frac{\partial^2 w}{\partial x^2} + T_{22}\frac{\partial^2 w}{\partial y^2}\right) - c_0^2\left(T_{22}\frac{\partial \phi_y}{\partial y} + T_{12}\frac{\partial \phi_x}{\partial x}\right)\right] \\
&-\frac{\partial^2}{\partial x \partial y}\left[2b_0^2 T_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y}\right) - 2c_0^2 T_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)\right] \\
&-N_{xx}\frac{\partial^2 w}{\partial x^2} - 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} - N_{yy}\frac{\partial^2 w}{\partial y^2}\Bigg\}\delta w \Bigg\} dxdy
\end{aligned}
$$

$$
\begin{aligned}
&+\oint_\Gamma \Bigg\{\Bigg\{\left[(1+S_1-2S_2)\left(D_{11}\frac{\partial\phi_x}{\partial x}+D_{12}\frac{\partial\phi_y}{\partial y}\right)+(S_1-S_2)\left(D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}\right)\right]n_x\\
&\quad+\left[(1+S_1-2S_2)\left(D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right)+(2S_1-2S_2)\left(D_{66}\frac{\partial^2 w}{\partial x\partial y}\right)\right]n_y\Bigg\}\delta\phi_x\\
&\quad+\Bigg\{\left[(1+S_1-2S_2)\left(D_{22}\frac{\partial\phi_y}{\partial y}+D_{12}\frac{\partial\phi_x}{\partial x}\right)+(S_1-S_2)\left(D_{22}\frac{\partial^2 w}{\partial y^2}+D_{12}\frac{\partial^2 w}{\partial x^2}\right)\right]n_y\\
&\quad+\left[(1+S_1-2S_2)\left(D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right)+(2S_1-2S_2)\left(D_{66}\frac{\partial^2 w}{\partial x\partial y}\right)\right]n_x\Bigg\}\delta\phi_y\\
&\quad+\Bigg\{\Bigg[S_3Gh\left(\frac{\partial w}{\partial x}+\phi_x\right)-\frac{\partial}{\partial x}S_1\left(D_{11}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial\phi_x}{\partial x}\right)+D_{12}\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial\phi_y}{\partial y}\right)\right)\\
&\quad+\frac{\partial}{\partial x}S_2\left(D_{11}\frac{\partial\phi_x}{\partial x}+D_{12}\frac{\partial\phi_y}{\partial y}\right)\Bigg]n_x\\
&\quad+\Bigg[S_3Gh\left(\frac{\partial w}{\partial y}+\phi_y\right)-\frac{\partial}{\partial y}S_1\left(D_{22}\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial\phi_y}{\partial y}\right)+D_{12}\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial\phi_x}{\partial x}\right)\right)\\
&\quad+\frac{\partial}{\partial y}S_2\left(D_{22}\frac{\partial\phi_y}{\partial y}+D_{12}\frac{\partial\phi_x}{\partial x}\right)\Bigg]n_y\\
&\quad-\frac{\partial}{\partial x}\left[S_1D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right)+S_2D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right]n_y\\
&\quad-\frac{\partial}{\partial y}\left[S_1D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right)+S_2D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right]n_x\\
&\quad-\left[\left(N_{xx}\frac{\partial w}{\partial x}n_x\right)+\left(N_{yy}\frac{\partial w}{\partial y}n_y\right)+\left(N_{xy}\frac{\partial w}{\partial y}n_x\right)+\left(N_{xy}\frac{\partial w}{\partial x}n_y\right)\right]\Bigg\}\delta w\\
&\quad+\Bigg\{\left[S_1D_{11}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial\phi_x}{\partial x}\right)+S_1D_{12}\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial\phi_y}{\partial y}\right)-S_2\left(D_{11}\frac{\partial\phi_x}{\partial x}+D_{12}\frac{\partial\phi_y}{\partial y}\right)\right]n_x\\
&\quad+\left[S_1D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right)-S_2D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right]n_y\Bigg\}\delta\frac{\partial w}{\partial x}\\
&\quad+\Bigg\{\left[S_1D_{22}\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial\phi_y}{\partial y}\right)+S_1D_{12}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial\phi_x}{\partial x}\right)-S_2\left(D_{22}\frac{\partial\phi_y}{\partial y}+D_{12}\frac{\partial\phi_x}{\partial x}\right)\right]n_y\\
&\quad+\left[S_1D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right)-S_2D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right]n_x\Bigg\}\delta\frac{\partial w}{\partial y}\Bigg\}ds=0
\end{aligned}
\tag{5.9}
$$

For the above expression to be kinematically admissible, either the integral expression or the variable must be zero. The variable being zero gives a trivial solution, while the field integral expression being zero gives the differential equations.

5.2 System of partial differential equations

Focusing now on the field integral in Eq. (5.9). By introducing the dimensionless stiffness parameters from Table 4.1 as shown in Eqs. (5.6), we obtain a system of three partial differential equations that describe both Shi's and Reddy's HSDTs. Collecting the terms corresponding to the variations of $\delta\phi_x$, $\delta\phi_y$ and δw in the field integral we obtain the following differential equations:

From $\delta\phi_x$

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left[(1+S_1-2S_2)\left(D_{11}\frac{\partial\phi_x}{\partial x}+D_{12}\frac{\partial\phi_y}{\partial y}\right)+(S_1-S_2)\left(D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}\right)\right]\\
&+\frac{\partial}{\partial y}\left[(1+S_1-2S_2)\left(D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right)+(2S_1-2S_2)\left(D_{66}\frac{\partial^2 w}{\partial x\partial y}\right)\right]\\
&\quad -S_3Gh\left(\frac{\partial w}{\partial x}+\phi_x\right)=0
\end{aligned}
\tag{5.10}
$$

From $\delta\phi_y$

$$
\begin{aligned}
&\frac{\partial}{\partial y}\left[(1+S_1-2S_2)\left(D_{22}\frac{\partial\phi_y}{\partial y}+D_{12}\frac{\partial\phi_x}{\partial x}\right)+(S_1-S_2)\left(D_{22}\frac{\partial^2 w}{\partial y^2}+D_{12}\frac{\partial^2 w}{\partial x^2}\right)\right]\\
&+\frac{\partial}{\partial x}\left[(1+S_1-2S_2)\left(D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right)+(2S_1-2S_2)\left(D_{66}\frac{\partial^2 w}{\partial x\partial y}\right)\right]\\
&\quad -S_3Gh\left(\frac{\partial w}{\partial y}+\phi_y\right)=0
\end{aligned}
\tag{5.11}
$$

From δw

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left[S_3 G h\left(\frac{\partial w}{\partial x}+\phi_x\right)\right] \\
&+\frac{\partial}{\partial y}\left[S_3 G h\left(\frac{\partial w}{\partial y}+\phi_y\right)\right] \\
&-\frac{\partial^2}{\partial x^2}\left[\left(S_1-S_2\right)\left(D_{11}\frac{\partial \phi_x}{\partial x}+D_{12}\frac{\partial \phi_y}{\partial y}\right)+S_1\left(D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}\right)\right] \\
&-\frac{\partial^2}{\partial y^2}\left[\left(S_1-S_2\right)\left(D_{22}\frac{\partial \phi_y}{\partial y}+D_{12}\frac{\partial \phi_x}{\partial x}\right)+S_1\left(D_{22}\frac{\partial^2 w}{\partial y^2}+D_{12}\frac{\partial^2 w}{\partial x^2}\right)\right] \\
&-\frac{\partial^2}{\partial x \partial y}\left[\left(2S_1-2S_2\right)D_{66}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)+4S_1 D_{66}\frac{\partial^2 w}{\partial x \partial y}\right] \\
&\qquad -N_{xx}\frac{\partial^2 w}{\partial x^2}-2N_{xy}\frac{\partial^2 w}{\partial x \partial y}-N_{yy}\frac{\partial^2 w}{\partial y^2}=0
\end{aligned}
\tag{5.12}
$$

This system of differential equations constitute the core of the unified framework that can describe several of the higher order shear deformation theories by introducing the related stiffness parameters.

5.3 Boundary conditions

The boundary conditions in HSDT will not be thoroughly discussed in this study. However it is worth noting that the entities in the boundary integral in Eq. (5.9) can be grouped into five sections:

$\delta\phi_x, \quad \delta\phi_y, \quad \delta w, \quad \delta\frac{\partial w}{\partial x} \quad$ and $\quad \delta\frac{\partial w}{\partial y}$.

This implies that there are five pairs of boundary conditions on each edge, as opposed to the three pairs we obtained in the Mindlin plate theory. According to Shi [10] the total differential order of the three differential equations in Reddy's and Shi's HSDTs is ten. Therefore, five boundary conditions for each edge of plates are expected.

By setting each of the five sections equal to zero we obtain that *either* the integral expression *or* the variable in the following five pairs needs to be specified on boundaries Γ of plates [10].

$$\oint_{\Gamma}\Bigg\{\Bigg\{\Bigg[(1+S_1-2S_2)\left(D_{11}\frac{\partial\phi_x}{\partial x}+D_{12}\frac{\partial\phi_y}{\partial y}\right)+(S_1-S_2)\left(D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}\right)\Bigg]n_x$$
$$+\Bigg[(1+S_1-2S_2)\left(D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right)+(2S_1-2S_2)\left(D_{66}\frac{\partial^2 w}{\partial x\partial y}\right)\Bigg]n_y\Bigg\}\delta\phi_x\Bigg\}ds=0 \tag{5.13}$$

$$\oint_{\Gamma}\Bigg\{\Bigg\{\Bigg[(1+S_1-2S_2)\left(D_{22}\frac{\partial\phi_y}{\partial y}+D_{12}\frac{\partial\phi_x}{\partial x}\right)+(S_1-S_2)\left(D_{22}\frac{\partial^2 w}{\partial y^2}+D_{12}\frac{\partial^2 w}{\partial x^2}\right)\Bigg]n_y$$
$$+\Bigg[(1+S_1-2S_2)\left(D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right)+(2S_1-2S_2)\left(D_{66}\frac{\partial^2 w}{\partial x\partial y}\right)\Bigg]n_x\Bigg\}\delta\phi_y\Bigg\}ds=0 \tag{5.14}$$

$$\oint_{\Gamma}\Bigg\{\Bigg\{\Bigg[S_3Gh\left(\frac{\partial w}{\partial x}+\phi_x\right)-\frac{\partial}{\partial x}S_1\left(D_{11}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial\phi_x}{\partial x}\right)+D_{12}\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial\phi_y}{\partial y}\right)\right)$$
$$+\frac{\partial}{\partial x}S_2\left(D_{11}\frac{\partial\phi_x}{\partial x}+D_{12}\frac{\partial\phi_y}{\partial y}\right)\Bigg]n_x$$
$$+\Bigg[S_3Gh\left(\frac{\partial w}{\partial y}+\phi_y\right)-\frac{\partial}{\partial y}S_1\left(D_{22}\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial\phi_y}{\partial y}\right)+D_{12}\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial\phi_x}{\partial x}\right)\right)$$
$$+\frac{\partial}{\partial y}S_2\left(D_{22}\frac{\partial\phi_y}{\partial y}+D_{12}\frac{\partial\phi_x}{\partial x}\right)\Bigg]n_y$$
$$-\frac{\partial}{\partial x}\Bigg[S_1D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right)+S_2D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\Bigg]n_y$$
$$-\frac{\partial}{\partial y}\Bigg[S_1D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right)+S_2D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\Bigg]n_x$$
$$-\Bigg[\left(N_{xx}\frac{\partial w}{\partial x}n_x\right)+\left(N_{yy}\frac{\partial w}{\partial y}n_y\right)+\left(N_{xy}\frac{\partial w}{\partial y}n_x\right)+\left(N_{xy}\frac{\partial w}{\partial x}n_y\right)\Bigg]\Bigg\}\delta w\Bigg\}ds=0 \tag{5.15}$$

$$\oint_{\Gamma}\Bigg\{\Bigg\{\Bigg[S_1D_{11}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial\phi_x}{\partial x}\right)+S_1D_{12}\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial\phi_y}{\partial y}\right)-S_2\left(D_{11}\frac{\partial\phi_x}{\partial x}+D_{12}\frac{\partial\phi_y}{\partial y}\right)\Bigg]n_x$$
$$+\Bigg[S_1D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right)-S_2D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\Bigg]n_y\Bigg\}\delta\frac{\partial w}{\partial x}\Bigg\}ds=0 \tag{5.16}$$

$$\oint_{\Gamma}\left\{\left\{\left[S_1D_{22}\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial\phi_y}{\partial y}\right)+S_1D_{12}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial\phi_x}{\partial x}\right)-S_2\left(D_{22}\frac{\partial\phi_y}{\partial y}+D_{12}\frac{\partial\phi_x}{\partial x}\right)\right]n_y\right.\right.$$

$$\left.\left.+\left[S_1D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right)-S_2D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right]n_x\right\}\delta\frac{\partial w}{\partial y}\right\}ds=0 \tag{5.17}$$

5.4 Describing Reddy's and Shi's theories by unified differential equations

In this section we will show that the system of differential equations in Eqs. (5.10), (5.11) and (5.12) can describe both Reddy's and Shi's theories by introducing the respective stiffness parameters from Table 4.1. We will prove the correspondence by investigating the differential equation coupled with $\delta\phi_x$ (Eq. (5.10)).

Reddy

The first differential equation of Reddy's theory can be presented as: (see [8])

From $\delta\phi_x$

$$\begin{aligned}
&\frac{\partial}{\partial x}\left[\frac{4}{5}\left(D_{11}\frac{\partial\phi_x}{\partial x}+D_{12}\frac{\partial\phi_y}{\partial y}\right)-\frac{1}{5}\left(D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}\right)\right.\\
&\left.-\frac{4}{3h^2}\left[\frac{4h^2}{35}\left(D_{11}\frac{\partial\phi_x}{\partial x}+D_{12}\frac{\partial\phi_y}{\partial y}\right)-\frac{h^2}{28}\left(D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}\right)\right]\right]\\
&+\frac{\partial}{\partial y}\left[\frac{4}{5}\left(D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right)-\frac{2}{5}\left(D_{66}\frac{\partial^2 w}{\partial x\partial y}\right)\right.\\
&\left.-\frac{4}{3h^2}\left[\frac{4h^2}{35}\left(D_{66}\frac{\partial\phi_x}{\partial y}+D_{66}\frac{\partial\phi_y}{\partial x}\right)-\frac{2h^2}{28}D_{66}\frac{\partial^2 w}{\partial x\partial y}\right]\right]\\
&-\frac{2}{3}Gh\left(\frac{\partial w}{\partial x}+\phi_x\right)+\frac{4}{h^2}\frac{h^2}{30}Gh\left(\frac{\partial w}{\partial x}+\phi_x\right)=0
\end{aligned} \tag{5.18}$$

When multiplying out Eq. (5.18) we get

$$\frac{\partial}{\partial x}\left[\frac{68}{105}\left(D_{11}\frac{\partial\phi_x}{\partial x}+D_{12}\frac{\partial\phi_y}{\partial y}\right)-\frac{16}{105}\left(D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}\right)\right]$$
$$+\frac{\partial}{\partial y}\left[\frac{68}{105}\left(D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right)-\frac{32}{105}\left(D_{66}\frac{\partial^2 w}{\partial x\partial y}\right)\right]$$
$$-\frac{8}{15}Gh\left(\frac{\partial w}{\partial x}+\phi_x\right)=0 \tag{5.19}$$

The dimensionless stiffness parameters in Reddy's theory are

$$S_1=\frac{1}{21},\qquad S_2=\frac{1}{5},\qquad S_3=\frac{8}{15} \tag{5.20}$$

When introducing (5.20) into Eq. (5.10) we obtain

From $\delta\phi_x$

$$\frac{\partial}{\partial x}\left[\left(1+\frac{1}{21}-2\cdot\frac{1}{5}\right)\left(D_{11}\frac{\partial\phi_x}{\partial x}+D_{12}\frac{\partial\phi_y}{\partial y}\right)+\left(\frac{1}{21}-\frac{1}{5}\right)\left(D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}\right)\right]$$
$$+\frac{\partial}{\partial y}\left[\left(1+\frac{1}{21}-2\cdot\frac{1}{5}\right)\left(D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right)+\left(2\cdot\frac{1}{21}-2\cdot\frac{1}{5}\right)\left(D_{66}\frac{\partial^2 w}{\partial x\partial y}\right)\right]$$
$$-\frac{8}{15}Gh\left(\frac{\partial w}{\partial x}+\phi_x\right)=0 \tag{5.21}$$

which equals Eq. (5.19).

Shi

The first differential equation of Shi's theory can be presented as: (see [10])

From $\delta\phi_x$

$$\frac{\partial}{\partial x}\left[\frac{85}{84}\left(D_{11}\frac{\partial\phi_x}{\partial x}+D_{12}\frac{\partial\phi_y}{\partial y}\right)+\frac{5}{4}\cdot\frac{1}{105}\left(D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}\right)\right]$$
$$+\frac{\partial}{\partial y}\left[\frac{85}{84}D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)+\frac{5}{2}\cdot\frac{1}{105}\left(D_{66}\frac{\partial^2 w}{\partial x\partial y}\right)\right]$$
$$-T_{44}\left(\frac{\partial w}{\partial x}+\phi_x\right)=0 \tag{5.22}$$

The dimensionless stiffnessparameters i Shi's theory are

$$S_1 = \frac{1}{84}, \qquad S_2 = 0, \qquad S_3 = \frac{5}{6} \tag{5.23}$$

When introducing (5.23) into Eq. (5.10) we obtain

<u>From $\delta\phi_x$</u>

$$\begin{aligned}
&\frac{\partial}{\partial x}\left[\left(1+\frac{1}{84}-0\right)\left(D_{11}\frac{\partial\phi_x}{\partial x}+D_{12}\frac{\partial\phi_y}{\partial y}\right)+\left(\frac{1}{84}-0\right)\left(D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}\right)\right]\\
&+\frac{\partial}{\partial y}\left[\left(1+\frac{1}{84}-0\right)\left(D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right)+\left(2\cdot\frac{1}{84}-0\right)\left(D_{66}\frac{\partial^2 w}{\partial x\partial y}\right)\right]\\
&\quad-\frac{5}{6}Gh\left(\frac{\partial w}{\partial x}+\phi_x\right)=0
\end{aligned} \tag{5.24}$$

which equals Eq. (5.22).

The same goes for the other differential equations, and we can conclude that the equations derived in Eqs. (5.10), (5.11) and (5.12) in fact represents both theories.

5.5 Solving the HSDT buckling problem

We consider the same isotropic plate under uniform in-plane compressive load on all edges as we did in Section 3.5, see Fig. 3.1.

The boundary conditions for the simply supported plate are shown in Fig. 5.1.

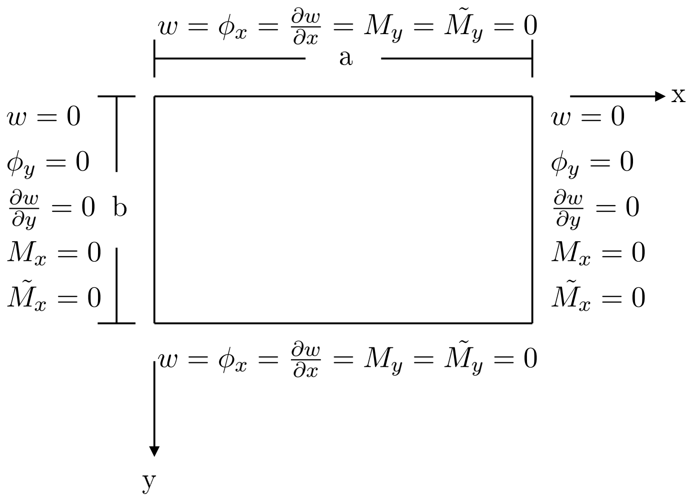

Figure 5.1: Boundary conditions for the simply supported plate in HSDT. $\tilde{M}_x$ and $\tilde{M}_y$ denote higher order moments.

The equation set found in Section 5.2 that solves the buckling problem is

$$\frac{\partial}{\partial x}\left[(1+S_1-2S_2)\left(D_{11}\frac{\partial \phi_x}{\partial x}+D_{12}\frac{\partial \phi_y}{\partial y}\right)+(S_1-S_2)\left(D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}\right)\right]$$
$$+\frac{\partial}{\partial y}\left[(1+S_1-2S_2)\left(D_{66}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)\right)+(2S_1-2S_2)\left(D_{66}\frac{\partial^2 w}{\partial x \partial y}\right)\right]$$
$$-S_3 G h\left(\frac{\partial w}{\partial x}+\phi_x\right)=0 \tag{5.25}$$

$$
\begin{aligned}
&\frac{\partial}{\partial y}\left[(1+S_1-2S_2)\left(D_{22}\frac{\partial \phi_y}{\partial y}+D_{12}\frac{\partial \phi_x}{\partial x}\right)+(S_1-S_2)\left(D_{22}\frac{\partial^2 w}{\partial y^2}+D_{12}\frac{\partial^2 w}{\partial x^2}\right)\right]\\
&+\frac{\partial}{\partial x}\left[(1+S_1-2S_2)\left(D_{66}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)\right)+(2S_1-2S_2)\left(D_{66}\frac{\partial^2 w}{\partial x\partial y}\right)\right]\\
&\quad -S_3Gh\left(\frac{\partial w}{\partial y}+\phi_y\right)=0
\end{aligned}
\tag{5.26}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left[S_3Gh\left(\frac{\partial w}{\partial x}+\phi_x\right)\right]\\
&+\frac{\partial}{\partial y}\left[S_3Gh\left(\frac{\partial w}{\partial y}+\phi_y\right)\right]\\
&-\frac{\partial^2}{\partial x^2}\left[(S_1-S_2)\left(D_{11}\frac{\partial \phi_x}{\partial x}+D_{12}\frac{\partial \phi_y}{\partial y}\right)+S_1\left(D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}\right)\right]\\
&-\frac{\partial^2}{\partial y^2}\left[(S_1-S_2)\left(D_{22}\frac{\partial \phi_y}{\partial y}+D_{12}\frac{\partial \phi_x}{\partial x}\right)+S_1\left(D_{22}\frac{\partial^2 w}{\partial y^2}+D_{12}\frac{\partial^2 w}{\partial x^2}\right)\right]\\
&-\frac{\partial^2}{\partial x\partial y}\left[(2S_1-2S_2)\,D_{66}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)+4S_1D_{66}\frac{\partial^2 w}{\partial x\partial y}\right]\\
&\quad -N_{xx}\frac{\partial^2 w}{\partial x^2}-N_{yy}\frac{\partial^2 w}{\partial y^2}=0
\end{aligned}
\tag{5.27}
$$

The Fourier series expansion method proposed by Navier can solve the buckling problem. The trial functions satisfying the boundary conditions take the form

$$w(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn}sin\left(\frac{m\pi x}{a}\right)sin\left(\frac{n\pi y}{b}\right) \tag{5.28a}$$

$$\phi_x(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}X_{mn}cos\left(\frac{m\pi x}{a}\right)sin\left(\frac{n\pi y}{b}\right) \tag{5.28b}$$

$$\phi_y(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Y_{mn}sin\left(\frac{m\pi x}{a}\right)cos\left(\frac{n\pi y}{b}\right) \tag{5.28c}$$

where W_{mn}, X_{mn} and Y_{mn} are series coefficients, m and n are positive integers.

Introducing the Fourier series into the equation set that solves the buckling problem gives the following matrix system

$$
\begin{bmatrix} C_1 & C_2 & C_3 \\ C_2 & C_4 & C_5 \\ C_3 & C_5 & C_6 \end{bmatrix} \begin{Bmatrix} X_{mn} \\ Y_{mn} \\ W_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{5.29}
$$

where the entities in the matrix of Eq. (5.29) are defined as

$$
\begin{aligned}
C_1 &= (1 + S_1 - 2S_2)\left(D_{11}\alpha^2 + D_{66}\beta^2\right) + S_3Gh \\
C_2 &= (1 + S_1 - 2S_2)\left(D_{12}\alpha\beta + D_{66}\alpha\beta\right) \\
C_3 &= (S_1 - S_2)\left(D_{11}\alpha^3 + D_{12}\alpha\beta^2 + 2D_{66}\alpha\beta^2\right) + S_3Gh\alpha \\
C_4 &= (1 + S_1 - 2S_2)\left(D_{22}\beta^2 + D_{66}\alpha^2\right) + S_3Gh \\
C_5 &= (S_1 - S_2)\left(D_{22}\beta^3 + D_{12}\alpha^2\beta + 2D_{66}\alpha^2\beta\right) + S_3Gh\beta \\
C_6 &= S_1\left(D_{11}\alpha^4 + 2D_{12}\alpha^2\beta^2 + D_{22}\beta^4 + 4D_{66}\alpha^2\beta^2\right) + S_3Gh\alpha^2 + S_3Gh\beta^2 - N\left(\alpha^2 + \beta^2\right)
\end{aligned} \tag{5.30}
$$

where $\alpha = \frac{m\pi}{a}$ and $\beta = \frac{n\pi}{b}$.

The nontrivial solution is found by setting the determinant of the matrix in Eq. (5.29) equal to zero.

$$
\begin{aligned}
&C_1C_4C_6 - C_1C_5^2 - C_2^2C_6 + C_2C_3C_5 + C_2C_3C_5 - C_3^2C_4 = 0 \\
&C_6(C_1C_4 - C_2^2) = C_1C_5^2 + C_3^2C_4 - 2C_2C_3C_5
\end{aligned} \tag{5.31}
$$

Inserting for C_6 and solving for N gives

$$
\begin{aligned}
N = &\Bigg(\frac{C_1C_5^2 + C_3^2C_4 - 2C_2C_3C_5}{(C_1C_4 - C_2^2)} \\
&- S_1\left(D_{11}\alpha^4 + 2D_{12}\alpha^2\beta^2 + D_{22}\beta^4 + 4D_{66}\alpha^2\beta^2\right) + T_{44}\alpha^2 + T_{55}\beta^2\Bigg)\frac{1}{\alpha^2 + \beta^2}
\end{aligned} \tag{5.32}
$$

The critical buckling load N_{cr} occurs at $n = 1$ while m can vary.

5.6 Relationship between HSDT and CPT

The superscripts K, H, M, R and S in this chapter denote *Kirchhoff*, *HSDT*, *Mindlin*, *Reddy* and *Shi* respectively.

In [11, p. 203] they present a buckling load relationship between Reddy's theory and CPT for a simply supported plate under uniform inplane forces. By following the same procedure as in [11] we have in this study found such a relationship valid for both Reddy's and Shi's HSDTs depending on the respective dimensionless stiffness parameters, (see Appendix A). We also believe that this relationship is valid for all HSDT's by introducing the respective dimensionless stiffness parameters. The relationship is given by

$$N^H = \frac{N^K\left(1 + \frac{N^K}{\frac{S_3}{S_1 - S_2^2}Gh}\right)}{1 + \frac{N^K}{\frac{S_3}{1+S_1-2S_2}Gh}} \tag{5.33}$$

Note that this relationship only holds for polygonal plates under uniform in-plane forces. The major advantage of such a relationship is that you can calculate the Kirchhoff buckling load first, and then use the formula to obtain the buckling load in HSDT. A similar relationship between FSDT and CPT is presented in [11], and it is given by

$$N^M = \frac{N^K}{1 + \frac{N^K}{\kappa Gh}} \tag{5.34}$$

where κ is the shear correction factor.

We can see from the relationships that the effect of shear deformation is to reduce the buckling load.

5.6.1 Introducing respective dimensionless stiffness parameters into relationship

The dimensionless stiffness parameters i Reddy's theory are

$$S_1 = \frac{1}{21}, \qquad S_2 = \frac{1}{5}, \qquad S_3 = \frac{8}{15} \tag{5.35}$$

Introducing (5.35) into Eq. (5.33) results in the this relationship between Reddy and CPT:

$$N^R = \frac{N^K\left(1 + \frac{N^K}{70Gh}\right)}{1 + \frac{N^K}{\frac{14Gh}{17}}} \tag{5.36}$$

The dimensionless stiffness parameters i Shi's theory are

$$S_1 = \frac{1}{84}, \qquad S_2 = 0, \qquad S_3 = \frac{5}{6} \tag{5.37}$$

Introducing (5.37) into Eq. (5.33) results in the this relationship between Shi and CPT, which is the same as between Reddy and CPT:

$$N^S = \frac{N^K \left(1 + \frac{N^K}{70Gh}\right)}{1 + \frac{N^K}{\frac{14Gh}{17}}} \tag{5.38}$$

So we see in fact that both Reddy's and Shi's theories yield the same buckling loads (Eq. (5.36) = Eq. (5.38)). Finding this relationship gives us a possibility to compare the results obtained by the Navier method, and it confirms that the two theories are equivalent when calculating the buckling loads, as will be shown in the next chapter.

Chapter 6

Buckling results

The buckling loads are calculated for the plate described in Sections 3.5 and 5.5 with varying plate geometry. In the following tables we present the buckling loads in CPT, FSDT and HSDT. In addition to the buckling loads calculated by using the Navier method, we present the buckling load found from the relationships in Eqs. (5.33) and (5.34). In all cases we apply

- $E = 2,1 \cdot 10^{11} N/m^2$
- $\nu = 0,3$
- $\kappa = \frac{5}{6}$ in FSDT

N_{cr} occurs at $n = 1$, $m = 1$ for all cases, meaning the plate buckles in one sine half wave in each direction.

The buckling loads in CPT for the isotropic case are calculated from

$$N^K = \frac{\pi}{b^2}\left(\frac{Ds^4m^4 + 2Ds^2m^2n^2 + Dn^4}{s^2m^2 + n^2}\right) \tag{6.1}$$

where $s = b/a$ is the plate aspect ratio. The buckling of Kirchhoff plates is widely presented in the literature. This formula is found from Wang et al. [13] (Wang et al. presents the same formula for the orthotropic case with biaxial loads).

6.1 Case A

Plate geometry:

- $a = 2,4m$
- $b = 1,6m$
- $h = 0,01m$, $h = 0,02m$ and $h = 0,03m$

Case A is obviously in the thin plate category and the results are expected to coincide regardless of which theory we use, since shear effects are not significant.

Table 6.1: Critical load $N_{cr}[N/m]$ in various theories.

Plate thickness [m]	Kirchhoff (CPT)	Mindlin (FSDT)	Reddy (HSDT)	Shi (HSDT)
$0,01$	107090	107080	107080	107080
$0,02$	856740	856190	856190	856190
$0,03$	2891500	2887400	2887400	2887400

Table 6.2: Critical load $N_{cr}[N/m]$ using relationship in Eqs. (5.33) and (5.34).

Plate thickness [m]	Kirchhoff (CPT)	Mindlin (FSDT)	Reddy/Shi (HSDT)
$0,01$	107090	107070	107070
$0,02$	856740	856200	856200
$0,03$	2891500	2887400	2887400

6.2 Case B

Plate geometry:

- $a = 1m$
- $b = 1m$
- $h = 0,1m$, $h = 0,2m$ and $h = 0,3m$

Case B is in the thick plate category and the neglection of shear deformation in CPT is expected to result in overestimation of N_{cr} in CPT.

Table 6.3: Critical load $N_{cr}[N/m]$ in various theories.

Plate thickness [m]	Kirchhoff (CPT)	Mindlin (FSDT)	Reddy (HSDT)	Shi (HSDT)
$0,1$	$379,6 \cdot 10^6$	$359,33 \cdot 10^6$	$359,35 \cdot 10^6$	$359,35 \cdot 10^6$
$0,2$	$3036,8 \cdot 10^6$	$2477,8 \cdot 10^6$	$2479,0 \cdot 10^6$	$2479,0 \cdot 10^6$
$0,3$	$10249,0 \cdot 10^6$	$6796,4 \cdot 10^6$	$6812,2 \cdot 10^6$	$6812,2 \cdot 10^6$

Table 6.4: Critical load $N_{cr}[N/m]$ using relationship in Eqs. (5.33) and (5.34).

Plate thickness [m]	Kirchhoff (CPT)	Mindlin (FSDT)	Reddy/Shi (HSDT)
$0,1$	$379,6 \cdot 10^6$	$359,33 \cdot 10^6$	$359,35 \cdot 10^6$
$0,2$	$3036,8 \cdot 10^6$	$2477,8 \cdot 10^6$	$2479,0 \cdot 10^6$
$0,3$	$10249,0 \cdot 10^6$	$6798,4 \cdot 10^6$	$6812,1 \cdot 10^6$

A steel plate one meter wide and $0,3$ meter thick is a very thick plate. The numerical results can be argued to be unrealistic, but the theoretical effect of the neglection of shear in CPT is well demonstrated.

6.3 Case C

Plate geometry:

- $a = 1m$
- $b = 0,5m$
- $h = 0,05m$, $h = 0,1m$ and $h = 0,15m$

Table 6.5: Critical load $N_{cr}[N/m]$ in various theories.

Plate thickness [m]	Kirchhoff (CPT)	Mindlin (FSDT)	Reddy (HSDT)	Shi (HSDT)
$0,05$	$118,63 \cdot 10^6$	$114,59 \cdot 10^6$	$114,59 \cdot 10^6$	$114,59 \cdot 10^6$
$0,1$	$949,0 \cdot 10^6$	$831,73 \cdot 10^6$	$831,9 \cdot 10^6$	$831,9 \cdot 10^6$
$0,15$	$3202,9 \cdot 10^6$	$2431,5 \cdot 10^6$	$2433,7 \cdot 10^6$	$2433,7 \cdot 10^6$

Table 6.6: Critical load $N_{cr}[N/m]$ using relationship in Eqs. (5.33) and (5.34).

Plate thickness [m]	Kirchhoff (CPT)	Mindlin (FSDT)	Reddy/Shi (HSDT)
$0,05$	$118,63 \cdot 10^6$	$114,59 \cdot 10^6$	$114,59 \cdot 10^6$
$0,1$	$949,0 \cdot 10^6$	$831,73 \cdot 10^6$	$831,9 \cdot 10^6$
$0,15$	$3202,9 \cdot 10^6$	$2431,5 \cdot 10^6$	$2433,7 \cdot 10^6$

In Table 6.7 we present the equivalent buckling load (Case C) obtained by introducing the dimensionless stiffness parameters of Touratier, Karama et al. and Mechab from Table 4.2. We can see that the various HSDTs give different values for the buckling loads, especially as the thickness increases.

Table 6.7: Critical load $N_{cr}[N/m]$ Touratier, Karama et al. and Mechab.

Plate thickness [m]	Touratier	Karama et al.	Mechab
$0,05$	$114,59 \cdot 10^6$	$114,61 \cdot 10^6$	$114,59 \cdot 10^6$
$0,1$	$832,09 \cdot 10^6$	$832,59 \cdot 10^6$	$831,97 \cdot 10^6$
$0,15$	$2435,1 \cdot 10^6$	$2438,3 \cdot 10^6$	$2433,9 \cdot 10^6$

The same buckling loads as those presented in table 6.7 are found by introducing the respective dimensionless stiffness parameters into the relationship formula in Eq. (5.33).

The plot in Figure 6.1 illustrates how CPT overestimates the critical buckling loads as the thickness of the plate in Case C increases. There are three graphs plotted in Figure 6.1:

- Blue graph: Kirchhoff
- Red graph: Mindlin
- Green dashed graph: Reddy/Shi

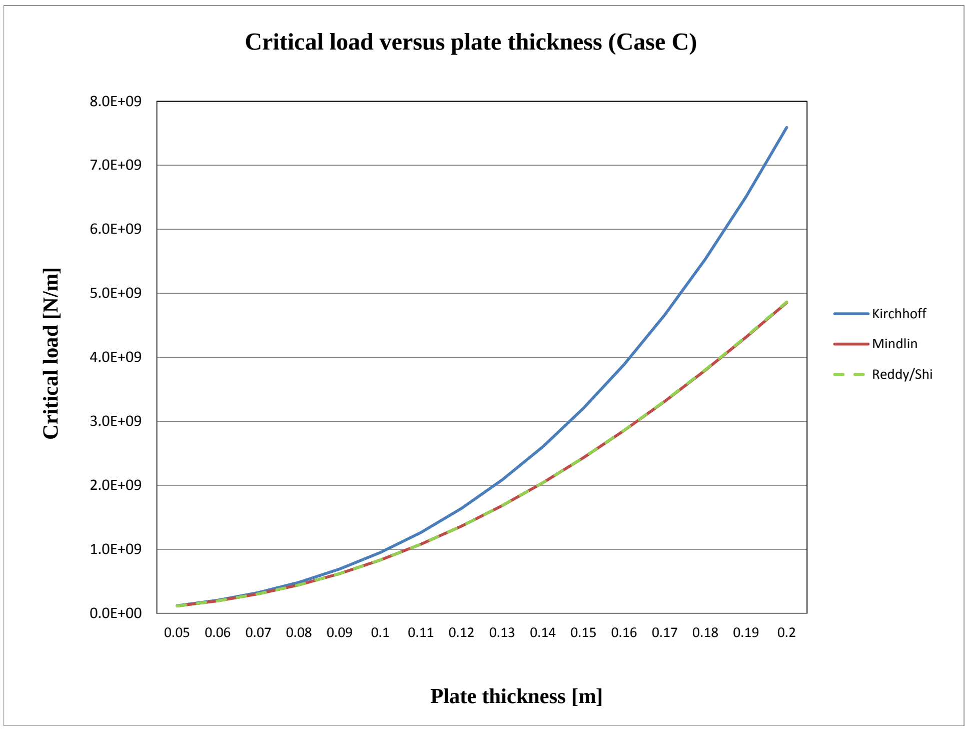

Figure 6.1: Illustration of the critical load estimated by CPT, FSDT and HSDT.

Figure 6.1 also illustrates that FSDT and HSDT predict virtually the same critical loads. A fact that demonstrates the statement by Reddy [9], that HSDTs should be used only when necessary in order to avoid the extra computational effort.

Chapter 7

Conclusion

7.1 Conclusion of results

The main result in this thesis is not the numerical results presented in Chapter 6, but rather the theory developed in Chapters 4 and 5 which is verified by the numerical results. It has been showed that several of the HSDTs presented in the literature can be described in a unified framework.

The unified HSDT is composed of three parts:

1. A higher order kinematic in-plane displacement field (4.2), based on a unified kinematics function. The unified kinematics function varies with the parameter ζ, which takes different value for the various HSDTs presented in the literature. By using the unified kinematics function, dimensionless values for the stiffness parameters in HSDT were calculated.
2. A system of variational consistent differential equations (Eqs. (5.10) to (5.12)) in terms of three displacement functions ϕ_x, ϕ_y, and w. The differential equations include the dimensionless stiffness parameters (S_1, S_2, and S_3), and by introducing the respective value for S_1, S_2, and S_3 it was proved that the differential equations in fact describe several of the HSDTs presented in the literature.
3. Five pairs of associated boundary conditions (Eqs. (5.13) to (5.17)) at each edge of plate boundaries.

The theory is validated through solving buckling problems for a simply supported plate under uniform in-plane loads. The buckling results for HSDT found by using the Navier method are compared with relationship formulas

(Eqs. (5.33) and (5.34)), which are valid for this type of buckling problems, in addition to equivalent buckling results estimated by CPT and FSDT. The buckling results illustrate the well known fact that CPT overestimates buckling loads for thick plates, and HSDT predicts virtually the same buckling loads as FSDT for the cases presented in this study.

The most surprising observation made, is the fact that the theories of Reddy and Shi predicts exactly the same buckling results. However, this does not imply that the theories are identical. In the study by Shi [10] he points out the differences between Reddy's and Shi's theories, and it is clearly illustrated in the present study that the two theories are distinct at the local stress level. For example, by investigating the stiffness parameters in the matrix system (Eq. 4.13) or in the differential equations (Eqs. (5.10) to (5.12)) the differences are obvious.

It has also been showed in this study that the investigated HSDTs can be presented as gradient elasticity FSDT models. This way of presenting plate theories is useful for a hierarchical classification of usual plate theories comprising Kirchhoff plate theory, Mindlin (and Reissner) plate theory and higher order theories, eqivalently showed for beam theories by Challamel [4].

7.2 Suggestions for further work

In order to clearly observe the benefits of HSDT compared to FSDT, the application of the present unified HSDT to the analysis of layered composite plates needs to be investigated. It would also be interesting to apply the present unified HSDT to buckling problems with other types of boundary conditions, and use other than the Navier method to solve buckling problems. In that respect, the expressions for the boundary conditions in Eqs. (5.13) to (5.17) need further investigation.

It is necessary to thoroughly verify that all HSDTs presented in the literature can be described by the unified HSDT. This can be done by following the same methodolgy as in this study on other HSDTs (Touratier, Mechab, Soldatos, Karama et al.), and compare numerical results with previously published results. For example, the study of Aydogdu [2] would be interesting in that respect.

Furthermore, it would be interesting to incorporate the von Karman nonlinear strains to achieve geometric nonlinear analysis of plates, and also to implement the present unified HSDT into finite element models.

References

[1] Altenbach, H., 2008.
Analysis of Homogenous and Non-Homogenous Plates.
de Borst, R. and Sadowski, T. (eds.), Lecture Notes on Composite Materials - Current Topics and Achievements. Springer Science+Business Media B.V., Netherlands.

[2] Aydogdu, M., 2006.
Comparison of Various Shear Deformation Theories for Bending, Buckling, and Vibration of Rectangular Symmetric Cross-ply Plate with Simply Supported Edges.
Journal of Composite Materials 40: 2143.

[3] Bergan, P.G., Syvertsen, T.G., 1989.
Knekning av søyler og rammer.
Tapir, Trondheim, 2. opplag.

[4] Challamel, N., 2011.
Higher-order shear beam theories and enriched continuum.
Mechanics Research Communications 38, 388-392.

[5] Challamel N., Mechab I., El Meiche N., Houari M.S.A., Ameur M., Ait Atmane H., 2012.
Buckling of generic higher-order shear beam/columns with elastic connections: local and nonlocal formulation.
J. Eng. Mech., in Press.

[6] Liu, Y., 2011.
A Refined Shear Deformation Plate Theory.
Int. J. for Computational Methods in Engineering Science and Mechanics. 12, 141-149.

[7] Reddy, J.N., 1997.
Mechanics of Laminated Composite Plates - Theory and Analysis.
CRC Press, Boca Raton, Florida.

[8] Reddy, J.N., Wang, C.M., 1998
Deflection relationships between classical and third-order plate theories.
Acta Mechanica. 130, 199-208.

[9] Reddy, J.N., 2004.
Mechanics of Laminated Composite Plates and Shells - Theory and Analysis.
CRC Press, Boca Raton, Florida, 2nd Edition.

[10] Shi, G., 2007.
A new simple third-order shear deformation theory of plates.
Int. J. Solids Struct. 44, 4299-4417.

[11] Wang, C.M., Reddy, J.N., Lee, K.H., 2000.
Shear Deformable Beams and Plates: Relationships with Classical Solutions.
Elsevier, Oxford, UK.

[12] Wang, C.M., Lim, G.T, Reddy, J.N., Lee, K.H., 2001.
Relationships between bending solutions of Reissner and Mindlin plate theories.
Engineering Structures 23, 838-849.

[13] Wang, C.M., Wang, C.Y., Reddy, J.N., 2005.
Exact Solutions for Buckling of Structural Members.
CRC Series in Computational Mechanics and Applied Analysis. Boca Raton, Florida.

[14] Yiang, Q.J., 2009.
Simplified Approaches to Buckling of Composite Plates.
Master Thesis at the University of Oslo.

Appendix A

Finding the relationship between HSDT and CPT

Due to the complexity of the following calculations they are presented in the appendix. The calculations can be difficult to follow and the reader is adviced to look at Wang et al. [11, p. 200] to see what is done there. The present calculations are based on the method presented in Wang et al., by transefering the calculations shown in [11] to the unified HSDT presented in this study. Some shortcuts are made at the end, but the calculations are presented here in any case, because the results are thought to be accurate and usefull in the context of the thesis.

We consider an isotropic plate under uniform in-plane compressive load N.

The three differential equations in Eqs. (5.10) to (5.12) can be presented as equilibrium equations in the following form:

$$\frac{\partial \tilde{M}_{xx}}{\partial x} + \frac{\partial \tilde{M}_{xy}}{\partial y} - \tilde{Q}_x = 0 \tag{A.1}$$

$$\frac{\partial \tilde{M}_{xy}}{\partial y} + \frac{\partial \tilde{M}_{yy}}{\partial y} - \tilde{Q}_y = 0 \tag{A.2}$$

$$
\begin{aligned}
&\frac{\partial \tilde{Q}_x}{\partial x} + \frac{\partial \tilde{Q}_y}{\partial y} \\
&- (S_1 - S_2) \frac{\partial^2}{\partial x^2} \left(D_{11} \frac{\partial \phi_x}{\partial x} + D_{12} \frac{\partial \phi_y}{\partial y} \right) \\
&- (S_1 - S_2) \frac{\partial^2}{\partial y^2} \left(D_{22} \frac{\partial \phi_y}{\partial y} + D_{12} \frac{\partial \phi_x}{\partial x} \right) \\
&- 2 (S_1 - S_2) \frac{\partial^2}{\partial x \partial y} D_{66} \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right) \\
&- S_1 \left(D_{11} \frac{\partial^4 w}{\partial x^4} + 2 D_{12} \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} \right) \\
&- N \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = 0 \qquad \text{(A.3)}
\end{aligned}
$$

The stress resultants marked with $\sim$ are the total stress resultants which can be read from the matrix system in Eq. (4.13).

When differentiating Eqs. (A.1) and (A.2), adding them together and using Eq. (A.3) we obtain

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} = N \nabla^2 w \qquad \text{(A.4)}$$

where we have introduced the laplace operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

We need to introduce the moment sum

$$M_s = \frac{M_{xx} + M_{yy}}{(1 + \nu)} \qquad \text{(A.5)}$$

From the matrix system in Eq. (4.13) in rows 1 and 2 we find

$$M_{xx} = D \phi_{x,x} + \nu D \phi_{y,y} - S_2 D (w_{,xx} + \phi_{x,x}) - S_2 \nu D (w_{,yy} + \phi_{y,y}) \qquad \text{(A.6a)}$$

$$M_{yy} = \nu D \phi_{x,x} + D \phi_{y,y} - S_2 \nu D (w_{,xx} + \phi_{x,x}) - S_2 D (w_{,yy} + \phi_{y,y}) \qquad \text{(A.6b)}$$

Introducing Eqs. (A.6) into Eq. (A.5) gives

$$M_s = D (1 - S_2) (\phi_{x,x} + \phi_{y,y}) - S_2 D \nabla^2 w \qquad \text{(A.7)}$$

$$(\phi_{x,x} + \phi_{y,y}) = \frac{M_s + S_2 D \nabla^2 w}{D (1 - S_2)} \qquad \text{(A.8)}$$

We also note that Eq. (A.4) can be expressed by the moment sum as

$$\nabla^2 M_s = N\nabla^2 w \tag{A.9}$$

The next thing we need is another expression for $(\phi_{x,x} + \phi_{y,y})$ which we find from Eq. (A.3) where we now express $\tilde{Q}_x$ and $\tilde{Q}_y$ as they are written in the differential equations (Eqs. (5.10) to (5.12)).

$$\begin{aligned}
S_3Gh\left(\frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_y}{\partial y}\right) &= \left(N - S_3Gh\right)\nabla^2 w \\
&+ \left(S_1 - S_2\right)\frac{\partial^2}{\partial x^2}\left(D_{11}\frac{\partial\phi_x}{\partial x} + D_{12}\frac{\partial\phi_y}{\partial y}\right) \\
&+ \left(S_1 - S_2\right)\frac{\partial^2}{\partial y^2}\left(D_{22}\frac{\partial\phi_y}{\partial y} + D_{12}\frac{\partial\phi_x}{\partial x}\right) \\
&+ 2\left(S_1 - S_2\right)\frac{\partial^2}{\partial x\partial y}D_{66}\left(\frac{\partial\phi_x}{\partial y} + \frac{\partial\phi_y}{\partial x}\right) \\
&+ S_1\left(D_{11}\frac{\partial^4 w}{\partial x^4} + 2D_{12}\frac{\partial^4 w}{\partial x^2\partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4}\right)
\end{aligned} \tag{A.10}$$

Considering isotropy and using the laplace operator, Eq. (A.10) reduces to

$$\begin{aligned}
S_3Gh\left(\frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_y}{\partial y}\right) &= \left(N - S_3Gh\right)\nabla^2 w \\
&+ \left(S_1 - S_2\right)D\nabla^2\left(\frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_y}{\partial y}\right) + S_1 D\nabla^4 w
\end{aligned} \tag{A.11}$$

Introducing Eq. (A.8) into the right hand side of Eq. (A.11) gives

$$\begin{aligned}
&\left(\frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_y}{\partial y}\right) = \\
&\left(\left(N - S_3Gh\right)\nabla^2 w + \frac{S_1 - S_2}{1 - S_2}\nabla^2 M_s + \frac{D\left(S_1 - S_2^2\right)\nabla^4 w}{1 - S_2}\right)\frac{1}{S_3Gh}
\end{aligned} \tag{A.12}$$

Introducing Eq. (A.12) back into Eq. (A.5) gives

$$\begin{aligned}
M_s =& D\left(1 - S_2\right)\left[\left(\frac{N - S_3Gh}{S_3Gh}\right)\nabla^2 w + \frac{S_1 - S_2}{\left(1 - S_2\right)S_3Gh}\nabla^2 M_s\right. \\
&\left. + D\frac{S_1 - S_2^2}{\left(1 - S_2\right)S_3Gh}\nabla^4 w\right] - S_2 D\nabla^2 w
\end{aligned} \tag{A.13}$$

Using Eq. (A.9) we can eliminate $\nabla^2 w$ from Eq. (A.13).

$$M_s = D\left(1-S_2\right)\left[\left(\frac{N-S_3Gh}{S_3Gh}\right)\frac{\nabla^2 M_s}{N}+\frac{S_1-S_2}{\left(1-S_2\right)S_3Gh}\nabla^2 M_s \right.$$
$$\left. D\frac{S_1-S_2^2}{\left(1-S_2\right)S_3Gh}\frac{\nabla^4 M_s}{N}\right]-S_2D\frac{\nabla^2 M_s}{N} \tag{A.14}$$

$$D^2\left(1-S_2\right)\frac{S_1-S_2^2}{\left(1-S_2\right)S_3Gh}\frac{\nabla^4 M_s}{N}$$
$$+\nabla^2 M_s\left[\frac{D\left(1-S_2\right)\left(N-S_3Gh\right)}{NS_3Gh}+\frac{D\left(1-S_2\right)\left(S_1-S_2\right)}{\left(1-S_2\right)S_3Gh}-\frac{S_2D}{N}\right]-M_s=0 \tag{A.15}$$

Eq. (A.15) is further processed until we obtain

$$\nabla^4 M_s+\nabla^2 M_s\frac{-S_3Gh}{\left(S_1-S_2\right)D}\left(1-\frac{1+S_1-2S_2}{S_3Gh}N\right)-\frac{NS_3Gh}{\left(S_1-S_2^2\right)D^2}M_s=0 \tag{A.16}$$

which can be expressed as

$$\left(\nabla^2+\lambda_1\right)\left(\nabla^2+\lambda_2\right)M_s=0 \tag{A.17}$$

where $(j=1,2)$.

$$\lambda_j=-\xi_1+(-1)^j\sqrt{\xi_1^2+\xi_2} \tag{A.18}$$

where ξ_1 and ξ_2 are the suitable expressions from Eq. (A.16).

We observe from Eq. (A.18) that λ_1 is negative and therefore does not lead to a feasible buckling solution. Thus, the buckling equation of the HSDT plate is governed by

$$\left(\nabla^2+\lambda_2\right)M_s=0 \tag{A.19}$$

λ_2 will have an expression which can be compared with an equivalent expression in the Kirchhoff theory, and we obtain the relationship between the buckling loads predicted by CPT and HSDT:

$$N^H = \frac{N^K \left(1 + \frac{N^K}{\frac{S_3}{S_1 - S_2^2} Gh}\right)}{1 + \frac{N^K}{\frac{S_3}{1 + S_1 - 2S_2} Gh}} \tag{A.20}$$

Appendix B

Matlab scripts

Kirchhoff

```
clear all
%Kirchhoff, simply supported under uniform in-plane forces%

%Material data (change MD for various cases)
E=2.1*1e11;
h=0.15;
v=0.3;
D=(E*h^3)/(12*(1-v^2));
a=1;
b=0.5;
s=b/a;

m=1; %(change m for various modes)%
n=1;

Nk=pi^2/b^2*((D*s^4*m^4+2*D*s^2*m^2*n^2+D*n^4)/(s^2*m^2+n^2)) %buckling load
```

Mindlin

```
clear all;
syms Nm real
%Mindlin, simply supported under uniform in-plane forces%

%Material data (change MD for various cases)
E=2.1*1e11;
v=0.3;
h=0.15;
a=1;
b=0.5;
D11=(E*h^3)/(12*(1-v^2));
D12=v*D11;
D66=D11*(1-v)/2;
k=5/6;
```

```
K=(k*E*h)/(2*(1+v));
n=1;

for m=1:4
alpha=m*pi/a;
beta=n*pi/b;

%C values
C1=-D11*alpha^2-D66*beta^2-K;
C2=-D12*alpha*beta-D66*alpha*beta;
C3=-K*alpha;
C4=-D11*beta^2-D66*alpha^2-K;
C5=-K*beta;
%C6=Nm*alpha^2+alpha*C3+beta*C5;

% Critical load
Nm=(((C1*(C5)^2+C4*(C3)^2-2*C2*C3*C5)/(C1*C4-(C2)^2))-(C3*alpha)-(beta*C5))/(alpha^2+beta^2); %buckling load
if m==1
    Nm1=Nm
elseif m==2
    Nm2=Nm
elseif m==3
    Nm3=Nm
else
    Nm4=Nm
end

end
```

HSDT

```
clear all;
syms Nh real
%HSDT, simply supported under uniform in-plane forces%

%Material data (change MD for various cases)
E=2.1*1e11;
v=0.3;
h=0.15;
a=1;
b=0.5;
D11=(E*h^3)/(12*(1-v^2));
D22=D11;
D12=v*D11;
D66=D11*(1-v)/2;

%S values. Change for various HSDT%
%Reddy%
S1=1/21;
S2=1/5;
S3=8/15;

%Shi%
%S1=1/84;
%S2=0;
%S3=5/6;
```

```
%Touratier%
%S1=5873/98123;
%S2=35208/155813;
%S3=1/2;

%Karama et al%
%S1=9474/128279;
%S2=25283/100044;
%S3=16822/35935;

%Mechab%
%S1=12007/311296;
%S2=5380/30127;
%S3=21999/39088;

K=(S3*E*h)/(2*(1+v));
n=1;

for m=1:4
alpha=m*pi/a;
beta=n*pi/b;


%C values
C1 = (1+S1-2*S2)*(-D11*alpha^2-D66*beta^2)-K;
C2 = (1+S1-2*S2)*(-D12*alpha*beta-D66*alpha*beta);
C3 = (S1-S2)*(-D11*alpha^3-D12*alpha*beta^2-2*D66*alpha*beta^2)-K*alpha;
C4 = (1+S1-2*S2)*(-D22*beta^2-D66*alpha^2)-K;
C5 = (S1-S2)*(-D22*beta^3-D12*alpha^2*beta-2*D66*alpha^2*beta)-K*beta;
%C6 = S1*(-D11*alpha^4-2*D12*alpha^2*beta^2-D22*beta^4-4*D66*alpha^2*beta^2)-K*alpha^2-K*beta^2+Nr*alpha^2;

% Critical load
Nh=(((C1*(C5)^2+(C3)^2*C4-2*C2*C3*C5)/(C1*C4-(C2)^2))
-(S1*(-D11*alpha^4-2*D12*alpha^2*beta^2-D22*beta^4-4*D66*alpha^2*beta^2)
-K*alpha^2-K*beta^2))/(alpha^2+beta^2); %buckling load

if m==1
    Nh1=Nh
elseif m==2
    Nh2=Nh
elseif m==3
    Nh3=Nh
else
    Nh4=Nh
end

end
```

Relationship formulas

```
clear all
%Relationship formulas. Change MD and NK for various cases%

%Material data%
E=2.1*1e11;
v=0.3;
G=E/(2*(1+v));
```

```
h=0.1;
k=5/6;

NK= 9.4900e+008;

%S values. Change for various HSDTs%
%Reddy%
S1=1/21;
S2=1/5;
S3=8/15;

%Shi%
%S1=1/84;
%S2=0;
%S3=5/6;

%Touratier%
%S1=5873/98123;
%S2=35208/155813;
%S3=1/2;

%Karama et al.%
%S1=9474/128279;
%S2=25283/100044;
%S3=16822/35935;

%Mechab%
%S1=12007/311296;
%S2=5380/30127;
%S3=21999/39088;

NH=NK*(1+NK*(S1-S2^2)/(S3*G*h))/(1+NK*(1+S1-2*S2)/(S3*G*h)) %Buckling load HSDT%

NM=NK/(1+NK/(k*G*h)) %Buckling load FSDT%
```